Proceedings of the ASME 2014 International Design Engineering Technical Conferences &
Computers and Information in Engineering Conference
IDETC/CIE 2014
August 17-20, 2014, Buffalo, New York, USA

DETC2014-34709

ON THE INTERPRETATION OF THE LAGRANGE MULTIPLIERS IN THE CONSTRAINT FORMULATION OF CONTACT PROBLEMS; OR WHY ARE SOME MULTIPLIERS ALWAYS ZERO?

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ABSTRACT

One method for modeling idealized contact between two bodies in mechanical system is based on the constraint approach, where Lagrange multipiers are introduced, which serve as constraint forces. In the usage of this formulation, there exists a linear dependancy between the Lagrange multipliers. Moreover, it has been observed that some Lagrange multipliers are always identical to zero. This sort of contradicts the basic notion that Lagrange multipliers in mechanical systems act as constraint forces which, when constraints are violated, push the system back in the desired configuration. In this paper it will be shown, by theory and example, that the above-mentioned linear dependency of the Lagrange multipliers, together with specific entries in the Jacobian of the constraint equations, results in some Lagrange multipliers being identical to zero.

1 INTRODUCTION

There are various ways to model contact between two bodies in multibody dynamic simulation, see for instance Pfeiffer & Glocker [1] and Shabana *et al.* [2]. When the bodies are assumed rigid, a constraint approach can be used. In this approach, the point of contact on each surface is then described by so-called surface parameters. Contact is expressed by having the two contact points coincide and that the two surfaces must have the same tangent planes at the contact point. These conditions form the nonlinear kinematic constraints. The constraints are then expressed in terms of the generalized coordinates of the bodies, the surface parameters and some geometrical constants, depending on the specific surfaces. The constraints are then added to the equations of motion by so-called Lagrange multipliers, as many as there are constraints. However, the surface parameters have no physical interpretation and therefore no inertia forces or applied forces associated with. Therefore the Lagrange multipliers, which act as constraint forces in the constraint equations of motion, show a linear dependency, as many as there are surface parameters [2].

In the usage of this formulation it has been noted [1] that some Lagrange multipliers are always identical to zero. This sort of contradicts the basic notion that Lagrange multipliers in mechanical systems act as constraint forces which, when constraints are violated, push the system back in the desired configuration.

In this paper it will be shown, by simple example, that the above-mentioned linear dependency of the Lagrange multipliers, together with specific entries in the Jacobian of the constraint equations, results in some Lagrange multipliers being identical to zero.

The paper is organised as follows. After this gentle introduction a general two-dimensional approach will be given after which results from a circular disk rolling on a sinusoidal road profile will be presented and discussed as an example. The paper ends with some conclusions.

[1] Private communication with José Escalona and J.P. Meijaard

2 CONSTRAINT CONTACT FORMULATION IN 2D

To simplify matters, instead of looking at the general three-dimensional problem, the constraint approach on a general two-dimensional rigid contact problem will be described in detail. There is no essential difference between the two- and the three-dimensional case. However, the two-dimensional case illustrates the issues somewhat easier. Details on the general three-dimensional approach can be found in for instance Shabana [2].

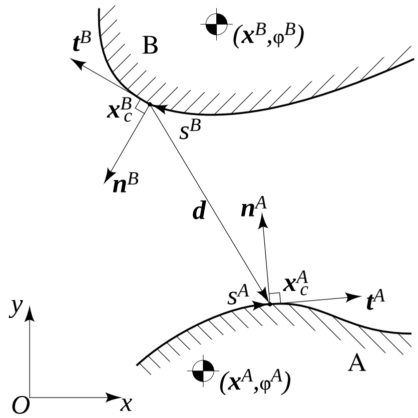

FIGURE 1. Two planar rigid bodies A and B in an inertial reference frame *O-xy* together with their potential contact point $\mathbf{x}_c^A$ and $\mathbf{x}_c^B$, and their pair of tangent and normal vectors $\mathbf{t}^A, \mathbf{n}^A$ and $\mathbf{t}^B, \mathbf{n}^B$ to the surface in the contact points.

Let us assume we have two rigid planar bodies, A and B, being in contact which each other. The two bodies in contact then share the same position and tangent in the contact point. In the constraint contact formulation we first define the contact point on each body i, $\mathbf{x}_c^i$ and then add the contact constraints. The position of the contact point is described by the position and orientation of the body, $(\mathbf{x}^i, \varphi^i)$, together with a contour parameter s^i, which describes the relative position of the contact point on the contour, as shown in Figure 1. So the functional dependancy of the contact point on each body i is now $\mathbf{x}_c^i = \mathbf{x}_c^i(\mathbf{x}^i, \varphi^i, s^i)$. The tangent at the contact point is defined as the partial derivative of the contact point with respect to the contour parameter,

$$\mathbf{t}^i = \partial \mathbf{x}_c^i / \partial s^i. \tag{1}$$

Note that this tangent, in general, does not have unit length. The normal to the contour at the the contact point is then defined from the tangent by a 90 degree rotation about the z-axis,

$$\mathbf{n}^i = \mathbf{R}\mathbf{t}^i \quad \text{with} \quad \mathbf{R} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{2}$$

The three contact constraints can now be defined as follows. With the relative distance $\mathbf{d} = \mathbf{x}_c^A - \mathbf{x}_c^B$ between the two contact points, sharing the same point at contact can be expressed by the two constraints,

$$c_n = \mathbf{d}^T \mathbf{n}^A \quad \text{and} \quad c_t = \mathbf{d}^T \mathbf{t}^A. \tag{3}$$

We choose to project the distance $\mathbf{d}$ along the normal and the tangent on body A so we have a clear physical interpretation of the constraint together with the Lagrange multiplier which acts as the constraint force. If c_n is violated then this is the normal indention between body A and B, and the Langrange multiplier associated with this constraint is then the normal contact force between body A and B. Likewise for c_t, which is the relative tangent displacement and the associated Lagrange multiplier is the tangent contact force between body A and B. Note that when $\mathbf{n}^A$ and $\mathbf{t}^A$ have no unit length these displacements and forces are scaled, but this scaling is immaterial for now. The third constraint is the condition that the two bodies A and B are tangent at the contact point,

$$c_p = {\mathbf{n}^A}^T \mathbf{t}^B. \tag{4}$$

We could also have used the normal of body B and the tangent of body A, which would give the same definition but for a sign change. Combining the three constraints in a vector $\mathbf{c} = [c_n, c_t, c_p]^T$, we can now form the constrained equations of motion for the system,

$$\begin{bmatrix} \mathbf{M} & \mathbf{0} & \mathbf{c}_{,\mathbf{q}}^T \\ \mathbf{0} & \mathbf{0} & \mathbf{c}_{,\mathbf{s}}^T \\ \mathbf{c}_{,\mathbf{q}} & \mathbf{c}_{,\mathbf{s}} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}} \\ \ddot{\mathbf{s}} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{Q} \\ \mathbf{0} \\ \mathbf{c2} \end{bmatrix}, \tag{5}$$

with the generalised coordinates $\mathbf{q}$, describing the position and orientation of the bodies, the surface or contour parameters $\mathbf{s} = [s^A, s^B]^T$, the Lagrange multipliers $\boldsymbol{\lambda} = [\lambda_n, \lambda_t, \lambda_p]^T$, the applied generalized forces $\mathbf{Q}$, the Jacobian matrices of the constraints $\mathbf{c}_{,\mathbf{q}} = \partial \mathbf{c}/\partial \mathbf{q}$ and $\mathbf{c}_{,\mathbf{s}} = \partial \mathbf{c}/\partial \mathbf{s}$, the convective constraint terms $\mathbf{c2} = \mathbf{c}_{,\mathbf{qq}}\dot{\mathbf{q}}\dot{\mathbf{q}} + 2\mathbf{c}_{,\mathbf{qs}}\dot{\mathbf{q}}\dot{\mathbf{s}} + \mathbf{c}_{,\mathbf{ss}}\dot{\mathbf{s}}\dot{\mathbf{s}}$ and the system mass matrix $\mathbf{M}$. The surface parameters, which are there to describe the location of the contact point, have no physical interpretation and therefore have no mass or forces associated with them. As a result, part of the constrained equations of motion (5) are a set of equilibrium conditions on the Lagrange multipliers ,

$$\mathbf{c}_{,\mathbf{s}}^T \boldsymbol{\lambda} = \mathbf{0}. \tag{6}$$

For the two-dimensional case these are two equations linear in the three Lagrange multipliers $\boldsymbol{\lambda}$. With these equilibrium conditions we can write the three Lagrange multipliers as a function of one of them. Substitution into the first set of equations in the constraint equations of motion (5) shows that the two-dimensional contact problem only depends on the value of one Lagrange multiplier.

Next, we investigate the nature of the Lagrange multipliers. The equilibrium on the Lagrange multipliers (6) written out in components is,

$$\begin{bmatrix} \partial c_n/\partial s^A & \partial c_t/\partial s^A & \partial c_p/\partial s^A \\ \partial c_n/\partial s^B & \partial c_t/\partial s^B & \partial c_p/\partial s^B \end{bmatrix} \begin{bmatrix} \lambda_n \\ \lambda_t \\ \lambda_p \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{7}$$

The first column is the most interesting one. If we substitute the previous defined expressions and expand we find,

$$\begin{aligned} \partial c_n/\partial s^A &= (\partial \mathbf{d}^T/\partial s^A)\mathbf{n}^A + \mathbf{d}^T(\partial \mathbf{n}^A/\partial s^A) \\ &= (\partial \mathbf{x}_c^{A^T}/\partial s^A)\mathbf{n}^A - \mathbf{d}^T\mathbf{t}^A \\ &= \mathbf{t}^{A^T}\mathbf{n}^A - \mathbf{d}^T\mathbf{t}^A \\ &= 0 - c_t \end{aligned} \tag{8}$$

Likewise for the second component we find,

$$\begin{aligned} \partial c_n/\partial s^B &= (\partial \mathbf{d}^T/\partial s^B)\mathbf{n}^A + \mathbf{d}^T(\partial \mathbf{n}^A/\partial s^B) \\ &= -(\partial \mathbf{x}_c^{B^T}/\partial s^B)\mathbf{n}^A + 0 \\ &= -\mathbf{t}^{B^T}\mathbf{n}^A \\ &= -c_p \end{aligned} \tag{9}$$

Clearly, when the c_t and c_p constraints are fulfilled, the first column is a null column. Thus if we express the λ_t and λ_p in terms of λ_n, the normal contact force, we will always find that these are zero, independent of the value of the normal contact force. One exception is when the leading matrix is singular. To investigate this we expand the remaining terms from (7) which results in

$$\begin{bmatrix} -c_t & (\mathbf{t}^{A^T}\mathbf{t}^A + c_n) & -\mathbf{t}^{B^T}\mathbf{t}^A \\ -c_p & -\mathbf{t}^{B^T}\mathbf{t}^A & (\partial^2\mathbf{x}_c^{B^T}/\partial s^{B^2})\mathbf{n}^A \end{bmatrix} \begin{bmatrix} \lambda_n \\ \lambda_t \\ \lambda_p \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{10}$$

Clearly, only in some very special case is the leading matrix rank deficient. We will come back to that in the example.

3 EXAMPLE

As an example we consider a circular disk rolling on a sinusoidal road under the act of gravity, as depicted in Figure 2. The uniform disk with radius r, has a mass m, and a mass moment of inertia about the centre off mass of $I = 1/2mr^2$. The

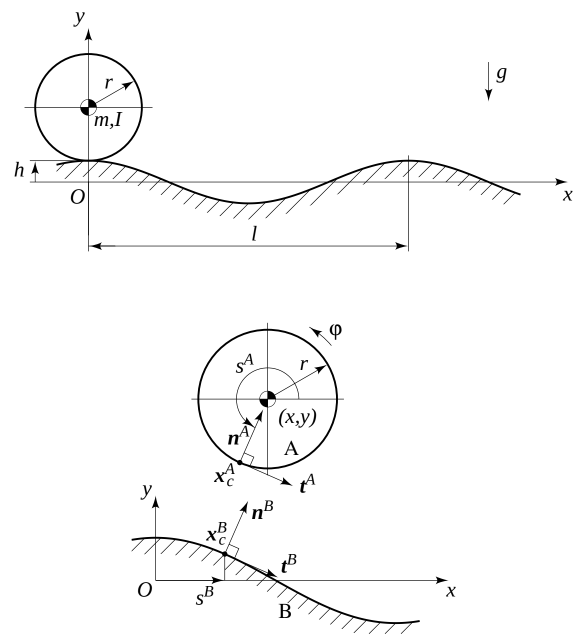

FIGURE 2. Planar disk rolling without slipping on a sinusoidal road profile under the action of gravity (top picture), together with the generalised coordinates (x, y, φ), surface parameters (s^A, s^B), contact points $(\mathbf{x}_c^A, \mathbf{x}_c^B)$, and tangent and normal vectors $(\mathbf{t}^A, \mathbf{n}^A)$ and $(\mathbf{t}^B, \mathbf{n}^B)$ used (bottom picture).

sinusoidal road profile is a cosine with wavelength l, and amplitude h. Gravity is acting in the minus y-direction with field strength g. With the space fixed road profile, the generalized coordinates $\mathbf{q}$ for this problem are the position of the centre of mass $\mathbf{x} = (x, y)^T$ and orientation φ of the disk with respect to the global reference frame; $\mathbf{q} = (x, y, \varphi)$. The contour of body A, the disk, is a circle with constant radius r. For the surface or contour parameter s^A, which locates the position of the contact point on the disk, we use the angle with respect to the global x-axis, counter clockwise. The surface or contour parameter s^B for body B, the space fixed sinusoidal road profile, is simply the x-coordinate of the curve. Accordingly, the expressions for the contact points on the disk A and the road profile B are,

$$\mathbf{x}_c^A = \begin{bmatrix} x \\ y \end{bmatrix} + r\begin{bmatrix} \cos(s^A) \\ \sin(s^A) \end{bmatrix} \quad \text{and} \quad \mathbf{x}_c^B = \begin{bmatrix} s^B \\ h\cos(2\pi s^B/l) \end{bmatrix}. \tag{11}$$

Besides the three contact constraints (3,4) we also need a constraint for the pure rolling condition. This is in general a non-holonomic constraint, a constraint expressed in the generalised speeds which can not be integrated. For the disk on the road profile this non-holonomic pure rolling constraint can be formulated from the notion that the tangential speed at the contact point should be zero,

$$c_s = \dot{\mathbf{x}}^T \mathbf{t}^A + r\dot{\varphi}. \tag{12}$$

Now the elements of the constraint equations of motion (5) can be formed. The mass matrix is $\mathbf{M} = \mathrm{diag}(m,m,I)$ and the Jacobian matrices can be formed by taking all the necessary derivatives. The convective terms **c2** are straightforward and will not be presented here. We will first focus on the interpretation of the Lagrange multipliers. Therefore, we write the equations of motion in the absence of accelerations, which are now in fact the static equilibrium conditions,

$$\begin{bmatrix} n_x^A & t_x^A & 0 & t_x^A \\ n_y^A & t_y^A & 0 & t_y^A \\ 0 & 0 & 0 & r \\ -c_t & r^2+c_n & T_1 & 0 \\ -c_p & T_1 & T_2 & 0 \end{bmatrix} \begin{bmatrix} \lambda_n \\ \lambda_t \\ \lambda_p \\ \lambda_s \end{bmatrix} = \begin{bmatrix} \sum f_x \\ \sum f_y \\ \sum M_\varphi \\ 0 \\ 0 \end{bmatrix}, \tag{13}$$

where we have introduced the terms $T_1 = -\mathbf{t}^{B^T}\mathbf{t}^A = r(\sin(s^A) + \omega h \sin(\omega s^B)\cos(s^A))$ and $T_2 = (\partial^2 \mathbf{x}_c^{B^T}/\partial s^{B^2})\mathbf{n}^A = rh\omega^2 \cos(\omega s^B)\sin(s^A)$, with $\omega = 2\pi/l$. The term T_1 will be of the order r and never be zero since this is the dot product of the two tangent vectors. The term T_2 clearly is the projected curvature, and can be zero. The first two equations in (13) can be rewritten in vector form as,

$$\sum \mathbf{f} = \lambda_n \mathbf{n}^A + (\lambda_t + \lambda_s)\mathbf{t}^A. \tag{14}$$

From this we can interpret $\lambda_n/|\mathbf{n}^A|$ as the normal contact force and $(\lambda_t + \lambda_s)/|\mathbf{t}^A|$ as the tangential contact force. Note that the Lagrange multiplier λ_p, the constraint force associated with the tangent curve condition, has no contribution to the forces and moment in the equations of motion, because the first three elements of the third column in (13) are all zero.

To illustrate the dependancy and zeroness of some of the Lagrange multipliers we use again the last two equations from (13), and rewrite these into,

$$\begin{bmatrix} r^2+c_n & T_1 \\ T_1 & T_2 \end{bmatrix} \begin{bmatrix} \lambda_t \\ \lambda_p \end{bmatrix} = \lambda_n \begin{bmatrix} c_t \\ c_p \end{bmatrix}. \tag{15}$$

Clearly, when the constraints on the tangential displacement and the tangent contours, c_t and c_p, are fulfilled the two Lagrange multipliers λ_t and λ_p are always zero, irrespective of the value of the normal contact force $\lambda_n/|\mathbf{n}^A|$. Only when the leading matrix is rank deficient can there be a non-zero solution. This is when there is non-conformal contact with identical curvature. Indeed, in that case there is no unique solution to the problem.

3.1 NUMERIC EXAMPLE

The above found results can be demonstrated by performing a numeric calculation in Matlab for the rolling disk on the sinusoidal road under the act of gravity. For parameters we use $r = 1, l = 5r, h = r/2, m = 1$ and $g = 10$, all in SI units. At $t = 0$ the disk is located at the apex in the origin and the centre of mass has a forward speed of $\dot{x} = 1$, together with and angular speed of $\dot{\varphi} = -1$.

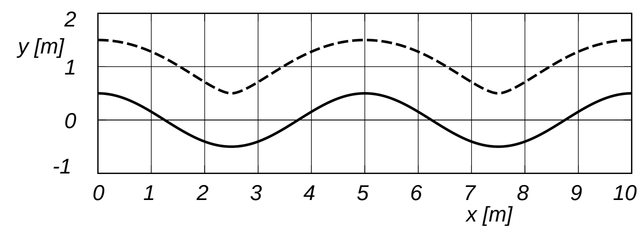

FIGURE 3. Path of the centre of mass of the rolling disk, dashed line, on a sinusoidal road profile together (cosine) road profile, solid line.

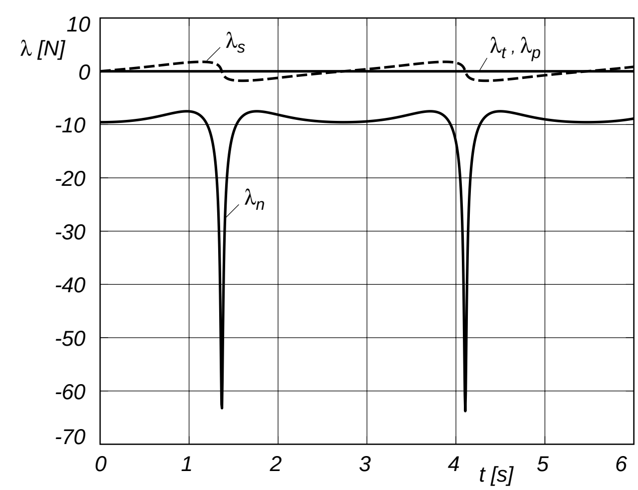

FIGURE 4. Lagrange multipliers as a function of time for the rolling disk on a sinusoidal road profile, with the normal contact force λ_n, solid curved line, the tangential contact force λ_s, dashed line, and the two zero valued multipliers λ_t and λ_p, solid horizontal straight lines.

Figure 3 show the path of the centre of mass of the disk rolling on the sinusoidal shaped road, together with the (cosine) road profile. Note the sharper curvature of the cm path down in the valleys compared to the one on top of the hills. Figure 4 shows the values of the four Lagrange multipliers as a function of time. The normal contact force, λ_n, which is the solid curved line in the graph shows a sharp increase when the disk approaches the bottom of the valley. The tangential contact force, λ_s, which is the dashed line, shows the acceleration and deceleration of the rotating disk when it travels from hill top to hill top. The solid horizontal straight lines are the other two Lagrange multipliers, λ_t and λ_p, which are essentially zero. In the numeric calculation scheme these show an error of max 10^{-12}, which is clearly within the calculating accuracy of the software used.

4 CONCLUSIONS

From a general two dimensional approach it has been shown in theory, what was already known in practice, that some Lagrange multipliers in constraint contact problems of rigid bodies are always zero.

ACKNOWLEDGMENT

Thanks to José Escalona for raising the question about the zero valued Lagrange multipliers in the constrained rigid body contact approach.

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