Available online at www.sciencedirect.com

ScienceDirect

Procedia Engineering 147 (2016) 907 – 911

Procedia Engineering

www.elsevier.com/locate/procedia

11th conference of the International Sports Engineering Association, ISEA 2016

A variational approach to determine the optimal power distribution for cycling in a time trial

Jenny de Jong[a], Robbert Fokkink[b,*], Geert Jan Olsder[b], A.L. Schwab[c]

[a]Utrecht University, Faculty of Mathematics, P.O. Box 80010, 3508TA Utrecht, The Netherlands
[b]Delft University of Technology, Faculty EEMCS, Mekelweg 4, 2628CD Delft, The Netherlands
[c]Delft University of Technology, BioMechanical Engineering, Mekelweg 2, 2628CD Delft, The Netherlands

Abstract

The optimal pacing strategy of a cyclist in an individual time-trial depends on terrain, weather conditions and the cyclists endurance capacity. Previous experimental and theoretical studies have shown that a suboptimal pacing strategy may have a substantial negative effect. In this paper we express the optimal pacing problem as a mathematical optimal control problem which we solve using Pontryagin's maximum principle. Our solution of the pacing problem is partly numerical and partly analytical. It applies to a straight course without bends. It turns out that the optimal pacing problem is a singular control problem. Intricate mathematical arguments are required to prove that the singular control times form a single interval: optimal pacing starts with maximum power and decays through a singular control, which may be degenerate, to minimum power.

Peer-review under responsibility of the organizing committee of ISEA 2016
Keywords: bicycling; power equation; maximum principle

1. Introduction

The aim of this study is to determine the optimal pacing strategy of a cyclist in an individual time trial, using variational methods. Previous studies [1,5,7] have shown that in time trials over a short distance, ranging from 1 to 4 km, the optimal strategy is an initial all out high power acceleration phase, followed by a lower constant power output. In these studies, a restricted set of different pacing strategies were compared numerically, and the optimal strategy was determined by direct computation. We introduce a different approach here, using Pontryagin's maximum principle to determine the optimal strategy from all possible pacing strategies. The solution turns out to involve a singular control, which is analogous to solutions of Goddard's problem in aerospace engineering [3].

1.1. The power equation

We derive the power equation that describes the propulsion of a cyclist in mathematical terms. The cyclist exerts a propulsive force F_P to overcome the resistance forces of air resistance F_A, slope resistance F_S, rolling resistance F_R, and bump resistance F_B. The excess, which may be negative, of F_P over the resistance will accelerate, or decelerate the cyclist. So we have that

$$F_P = F_A + F_S + F_R + F_B + F_{acc} \tag{1}$$

* Corresponding author. Tel.: +31-15-2789215; fax: +31-15-2787255
E-mail address: r.j.fokkink@tudelft.nl

Peer-review under responsibility of the organizing committee of ISEA 2016
doi:10.1016/j.proeng.2016.06.280

For time-trials in an approximately level terrain, air resistance is the main resistance force. It is given by $F_A = K_A(v + v_w)^2$ where v is the velocity of the rider, v_w is the velocity of the wind, and K_A is the drag coefficient. The slope resistance is $F_S = mg\sin\varphi$ where g is the gravitational acceleration and φ is the angle of inclination ($\tan\varphi$ is the slope). The rolling resistance is by $F_R = mgC_R$ where C_R is resistance coefficient. There is no general formula for the bump resistance, but we will assume that the terrain is smooth and neglect this factor. Finally, $F_{acc} = m_e a$ where m_e is the effective mass, which slightly exceeds the mass m of the rider plus the bike to account for the kinetic energy of the bicycle's rotating wheels. This all adds up to

$$F_P = K_A\left(v + v_w\right)^2 + mg\left(\sin\varphi + C_R\right) + m_e a \tag{2}$$

The power exerted by the rider on the system is given by $u(t) = F_P v(t)$ and rewriting the equation in these terms gives a differential equation, which is known as the *power equation* [8]:

$$u(t) = \left[K_A\left(v(t) + v_w\right)^2 + mg\left(\sin\varphi + C_R\right) + m_e\frac{dv(t)}{dt}\right]\cdot v(t) \tag{3}$$

To contain the mathematical details of our analysis, we simplify the equation by assuming that there is no headwind. It thus reduces to

$$u(t) = \left[c_1 v(t)^2 + c_2 + c_3\frac{dv(t)}{dt}\right]\cdot v(t) \tag{4}$$

1.2. The power model

The power output of the rider depends on aerobic and anaerobic energy sources. Aerobic energy is unlimited in capacity but limited in rate, and anaerobic energy is limited in capacity but allows a much larger rate. There exist many different models for power output, from which we adopt the model of Skiba et al [6], for which $CP \le u(t) \le u_{\max}$, where CP is the critical power level that is sustained by aerobic energy and $u_{\max}$ is the maximum power that the rider can achieve by use of anaerobic energy. Thus $u(t) - CP$ is excess energy and the model says that the total amount of excess energy that can be applied is a constant W, which depends on the rider. We consider a short time trial only, in which the rider cannot recharge the anaerobic energy. Thus we arrive at the problem of minimizing the total time T of the time-trial, under the constraints that $\int_0^T u(t) - CP dt$ is constant, that $CP \le u(t) \le u_{\max}$, and that the solution $v(t)$ of the power equation satisfies $\int_0^T v(t)dt = L$ where L is the length of the circuit. This is a mathematical control problem that fits into the framework of Pontryagin's maximum principle.

2. The time-trial control problem and its solution

The time-trial control problem is to minimize T subject to the constraint that the total excess power that is used up to that time is equal to a constant W and that the integral over the velocity $v(t)$ equals the length L of the circuit. To put this in a form that allows the application of Pontryagin's maximum principle, we rewrite the equations into an integral-differential form, as follows:

$$\max_{CP \le u(t) \le u_{\max}} \int_0^T -1\,dt$$

subject to the constraints

$$\left[\begin{aligned} \frac{dx_1}{dt} &= x_2(t), \\ \frac{dx_2}{dt} &= \frac{u(t)}{c_3 x_2(t)} - \frac{c_1 x_2(t)^2}{c_3} - \frac{c_2}{c_3}, \\ \frac{dx_3}{dt} &= u(t) - CP. \end{aligned}\right.$$

The boundary conditions are $\boldsymbol{x_1(0) = 0, x_2(T) = L, x_2(0) = \alpha > 0, x_3(0) = 0, x_3(T) = W}$. Note that we require that the initial velocity is positive (but can be arbitrarily small) to avoid a singularity in the constraint at time zero. The equations thus describe the power optimization of a rider that starts from a standstill or a small velocity who has to complete a straight course in minimal time. This is a minimization problem that fits into the framework of the maximum principle [2], which yields the Hamiltonian function

$$H(x,u,\lambda) = -1 + \lambda_1(t)x_2(t) + \lambda_2(t)\left[\frac{u(t)}{c_3 x_2(t)} - \frac{c_1 x_2(t)^2}{c_3} - \frac{c_2}{c_3}\right] + \lambda_3(t)\big(u(t) - CP\big) \tag{5}$$

The three constraints on the original problem combine with three adjoint equations on the multipliers, as follows

$$\begin{cases} \frac{dx_1}{dt} = x_2(t) & x_1(0) = 0, x_1(T) = L \\ \frac{dx_2}{dt} = \frac{u(t)}{c_3 x_2(t)} - \frac{c_1 x_2(t)^2}{c_3} - \frac{c_2}{c_3} & x_2(0) = \alpha > 0 \\ \frac{dx_3}{dt} = u(t) - CP & x_3(0) = 0, x_3(T) = W \\ \frac{d\lambda_1}{dt} = 0 & \\ \frac{d\lambda_2}{dt} = -\left(\lambda_1 - \frac{\lambda_2(t)u(t)}{c_3\big(x_2(t)\big)^2} - \frac{2c_1}{c_3}\lambda_2(t)x_2(t)\right) & \lambda_2(T) = 0 \\ \frac{d\lambda_3}{dt} = 0 & \end{cases} \tag{6}$$

Since the Hamiltonian (5) is linear in u, the optimal control u^* is of a simple form and it depends on the sign of

$$\frac{\lambda_2(t)}{c_3 x_2(t)} + \lambda_3(t) \tag{7}$$

If the sign in (7) it is negative, then $u^*=u_{\min}$. If the sign is positive then $u^*=u_{\max}$, and if the expression is equal to zero, then $u^*=u_{\sin}$, which is a constant singular power level in between $u_{\min}$ and $u_{\max}$. It is possible to prove that the expression in (7) decays in time. This implies that it is optimal for the rider to take off at a maximum power level, consolidate at an intermediate singular level for some time, and finish the race at minimal power. Despite this simple form of the optimal distribution, the mathematical analysis is involved and it is not trivial to establish that (7) decays in time. For a complete analysis we refer to [4].

3. An example

As an example we consider a 5 km time-trial with the following parameters: the initial velocity α equal to 1 *m/s*, total energy of W = 20,000 Joule, maximum power $u_{\max}$ = 800 Watt and critical power CP = 300 Watt. The constants in the power equation are c_1 = 0.128, c_2 = 3.924; c_3 = 78.1. These parameters were computed from c1 = $0.5C_dA\rho$, where we set the product of the drag coefficient and the frontal area equal to C_dA = 0.217 and ρ is the air density; $c_2 = mg(s+C_R)$ where we take slope s = 0 and C_R = 0.005 and we take $c_3 = m$ = 78. If the rider goes all out at maximum power, then W is depleted after roughly 40 seconds and the rider has covered approximately 1 kilometer. Therefore, the bang-bang control in which the rider falls back from $u_{\max}$ to $u_{\min}$ is optimal for this short time trial. For a slightly longer trial of 5 kilometers a bang-singular-bang control is optimal. The rider sustains the maximum power level for 10 seconds, reaching a velocity of 13 m/s, then switches back to the singular power level to sustain this velocity until the anaerobic energy runs out, and then in the last minute finishes the trial at critical power when the velocity decreases to 12 m/s, as can be sustained at critical power. This result is very similar to results of De Koning et al [5] for short time trials.

The singular power level can be computed from the adjoint equations, as follows. The ratio of λ_2 and x_2 is a constant γ during the singular interval and so it follows from equation (7) that

$$\frac{d}{dt}\left(\frac{\lambda_2}{x_2}\right) = 3\frac{c_1}{c_3}\gamma x_2(t) + \left(\frac{c_2}{c_3}\gamma - 1\right)\frac{1}{x_2(t)} = 0 \tag{8}$$

which implies that x_2 remains constant during the entire singular time interval (as can also be observed in Fig. 1), and hence so does the multiplier λ_2. From this we can compute the singular power level

$$u_{\sin} = \frac{(c_3 + 2c_2\gamma)\sqrt{\dfrac{c_3 - c_2\gamma}{c_1\gamma}}}{3\sqrt{3}\gamma} \tag{9}$$

The singular power depends on the three coefficients in the power model, but it also depends on x_2, and on the control λ_2 which needs to be computed from the adjoint equations. Our equation for $u_{\sin}$ depends on x_2, which on its turn is determined by the singular power level. This is an implicit expression and it is not easy to see how it varies with the parameters. It is of course possible to determine u_{sing} numerically and our computations show that $u_{\sin}$ approaches $u_{\max}$ in short trials and it approaches *CP* in long trials, as is to be expected. The rider starts out at a high acceleration, settles down in a velocity that can be maintained almost to the end of the trial, and then coasts to the finish in a final short time interval. The result remains qualitatively the same for longer time trials.

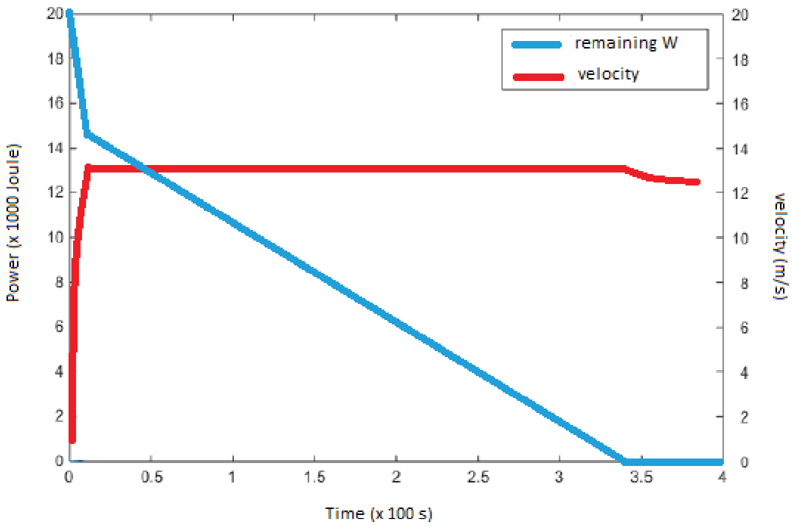

Fig. 1. Example of power output and energy distribution in a 5 km time trial. The blue line represents the remaining energy of the rider and the red line represents the velocity. Note that the rider starts with maximum power and after roughly 10 seconds switches to the singular power level, to sustain a constant velocity of about 13 *m/s*. Approximately one minute before reaching the finish line, the anaerobic energy *W* has run out and the rider coasts to the finish line.

4. Conclusions and acknowledgements

The main goal of our study was to determine whether optimal control theory can be applied to study time trials in the simplest case, where all physical parameters such as slope and air resistance are constant in time and the anaerobic energy level of the athlete cannot be recharged. It turns out that this is possible, but that solving the optimal control problem numerically is not trivial. In particular, solving the differential equation for the second multiplier λ_2 turned out to be intricate. By trial and error we found that the proper numerical approach is by a downwind scheme, iterating backwards in time. Matlab code for solving the differential equation (6) can be found in the appendix of [4].

To extend our results to a realistic situation, it is necessary to divide the circuit of the time-trial into short segments between bends. The velocity of the rider in a bend depends on the curvature and the width of the bend, which serve as boundary conditions on the adjoint equation. The rider must divide the anaerobic energy between the segments, which introduces another optimization aspect to the problem. Some preliminary investigations using linear programming, and analyzing a circuit that consists of five segments are given in [4].

Ideally, one would be able to build a real time optimal control system, that includes space varying parameters of the terrain, and time varying weather forecasts and physiological parameters of the athlete. In principle, developing such a system is possible. Future work is directed towards developing such a numerical simulation making use of contemporary optimalcontrol numerical tools.

We would like to thank Teun van Erp and others of the Giant-Alpecin cycling team for supplying data, selecting an appropriate power model and tuning its coefficients.

References

[1] Atkinson G, Brunskill A. Pacing strategies during a cycling time trial with simulated headwinds and tailwinds, Ergonomics 43 (10), 2000, 1449-1460, doi:10.1080/001401300750003899.

[2] Bryson AE, Ho Y. Applied Optimal Control: Optimization, Estimation and Control, CRC Press 1975, ISBN 000-0891162283

[3] A. Dmitruk, I. Samylovskiy, A simplied Goddard problem in the presence of a nonlinear media resistance and a bounded thrust, European Control Conference 2013, 341-346.

[4] de Jong J, The optimal power distribution for cycling in a time-trial, MSc thesis, Utrecht University, 2015.

[5] de Koning JJ, Bobbert MF, Foster C. Determination of optimal pacing strategy in track cycling with an energy flow model, J. Science Medic. Sport 2(3) 1999, 266-277, doi:10.1016/s1440-2440(99)80178-9

[6] Skiba PF, Jackman S, Clarke D, Vanhatalo A, Jones AM, Effect of work and recovery durations on W' reconstitution during intermittent exercise, Med. Sci. Sports Exerc. 46(7) 2014, 1433-1440, doi:10.1249/MSS.0000000000000226

[7] Underwood Jermy IM. Determining optimal pacing strategy for the track cycling individual pursuit event with a mixed energy mathematical model, Sports Eng. 17 (2014), 183-196, doi:10.1007/s12283-014-0153-3

[8] Wilson DG. Bicycling science, 3rd edition, Cambridge (MT), MIT press 2004, ISBN 978-0262731546.