Kinematics and Dynamics of Mechanisms

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A brief account is given of the finite element method, as it was developed as a tool for kinematic and dynamic analysis of mechanisms.

In the finite element method permanent contact between bodies is not expressed by kinematic contraints, but by letting them have boundary points in common. The dynamics of multi-body systems is derived on the basis of the principle of virtual power. Both rigidity and deformability are introduced as constitutive assumptions.

The ease, by which the deformability can be handled, leads to interesting possibilities of modelling elements like sliders and gears. In the computer-programs PLANAR and SPACAR full advantage is taken of these possibilities. Thus user friendly, general purpose computer programs have been developed, which fit in nicely with the finite element approach, that has proved to be so powerful in modelling and analysing complicated problems of mechanics.

A few examples illustrate the capabilities of the method, and of the computerprograms in their present state.

1. Introduction

Though presently the finite element method is usually put forward as a numerical method for the determination of approximate solutions for continuum problems, the method has been used in structural mechanics since the days of Eiffel. Finite elements may be looked upon as models of mechanical behaviour in their own right. This approach, that was developed so successfully for the analysis of the strength and stiffness of complex structures, has proved to be also particularly useful in the kinematic and dynamic analysis of mechanisms.

Classically the theory of kinematics and dynamics of mechanisms is developed for rigid bodies. Contact between these bodies is expressed by constraint equations. In the finite element method permanent contact between bodies is obtained by letting them have nodal points in common, and in these nodal points they share some or all the coordinates of these points.

bodies, in the finite element method we have to impose conditions on the deformation modes, determined by the number of nodal coordinates of each finite element and its number of degrees of freedom as a rigid body. The multiple rigid body situation is obtained by putting all deformation modes equal to zero.

The expressions for the rates of deformation represent linear mappings of the space of velocity components onto the space of rates of deformation. The null space of such a linear mapping determines at any moment the space of the degrees of freedom of the mechanism with rigid links. But because of the fact that we start out from expressions for the deformation modes as nonlinear functions of the coordinates, it is very easy to extend the analysis to the case of mechanisms with deformable links. Then constitutive equations for the deformations have to be supplied. They may express simply linear elastic behaviour, but by these constitutive equations we can also model actuators and other active elements in a mechanism.

The equations of motion for multi-body systems are derived by the principle of virtual power. If the generalized coordinates of the mechanism with rigid links together with possible modes of deformation are taken as the system parameters we obtain a nonlinear system of ordinary differential equations of second order. The expressions for the strains, which are determined as nonlinear functions of the nodal coordinates, are used as correctors in a predictor-corrector integration scheme.

The finite element theory for kinematics and dynamics of mechanisms was originated in [1]. Since this original publication computerprograms have been developed, that can handle the zero'th, the first, and the second order kinematics of plane and spatial mechanisms. These programs, PLANAR and SPACAR, are the product of a fruitful cooperation between the Laboratory for Production Automation and Mechanisms and the Laboratory of Engineering Mechanics of the Department of Mechanical Engineering of the TH-Delft. In the latter laboratory a program called DYNAMO was set up for the analysis of mechanisms with flexible links.

Some results obtained with the aid of these computerprograms will illustrate the potential of the finite element theory for the kinematics and dynamics of mechanisms.

2. Finite Element Representation of Mechanisms

Let us first consider the simplest possible link in a mechanism: a connecting rod between two hinges (Fig. 1). For motions in a plane the four coordinates of the endpoints of the link determine not only its position, but also its length.

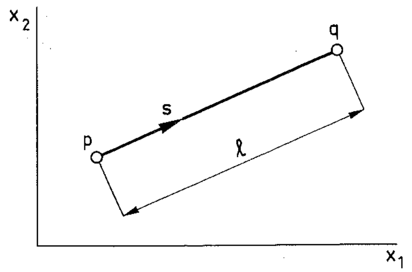


Fig. 1. Connecting rod between two hinges.

If the link is rigid the change of length must be equal to zero, which is the constraint condition for this link as part of a mechanism with rigid links. However we begin by defining explicitely the deformation mode of this link by

$$\varepsilon = [(x_1^q - x_1^p)^2 + (x_2^q - x_2^p)^2 - \ell_o^2]/(2\ell_o^2). \qquad (2.1)$$

The reason that we take the difference of the square of the lengths in the deformed and the undeformed state is that then we have only derivatives of ε with respect to the coordinates up to the second order.

Now we can model a simple four-bar linkage (Fig. 2).

For planar motion the nodal coordinates are elements of a eight-dimensional vectorspace. We may consider this space as the direct sum of the space of the coordinates of the moving nodes (x_k^c), and the space of the coordinates with fixed values (x_ℓ^o). With four bars we have four deformation parameters, defined as nonlinear functions of the coordinates.

$$\varepsilon_i = D_i(x_k^c, x_\ell^o), \qquad i = 1...4, \qquad k + \ell = 1...8. \qquad (2.2)$$

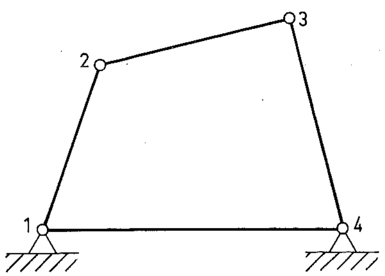


Fig. 2. Four-bar linkage.

If we take the bar between the nodes 1 and 4 as a base, then the elements of the vectors, x_k^c and x_ℓ^o, are given by

$$x^{cT} = |x_1^2\ x_2^2\ x_1^3\ x_2^3|,$$

$$x^{oT} = |x_1^1\ x_2^1\ x_1^4\ x_2^4|. \qquad (2.3)$$

For a motion with undeformable links at any moment the condition

$$D_{i,k}^c\ \dot{x}_k^c = \delta_{ij}\ \dot{\varepsilon}_j = \dot{\varepsilon}_i = 0 \qquad (2.4)$$

must be fulfilled. We may look upon the equations (2.4) as a linear mapping of the space of velocity components, $\dot{x}_k^c$, onto the space of rates of deformation. Undeformability of the links implies that for motion the velocity components must lie in the null space of the linear mapping.

Now we note, as a result of linear algebra theory, that for any linear mapping the matrices, that represent the mapping, can be given the following form by linear transformations and by interchanging rows as well as columns.

$$\begin{bmatrix} I & B^* \\ 0 & 0 \end{bmatrix} \dot{x}^c = \begin{bmatrix} D^{cc^{-1}} & 0 \\ -B^T & I \end{bmatrix} \dot{\varepsilon}. \qquad (2.5)$$

With the rows of

$$Z^{*T} = [-B^{*T} \quad I] \qquad (2.6)$$

we have a basis for the null space of the linear mapping, transformed to the space of velocity components. With the aid of

$$\Sigma^o = \begin{bmatrix} D^{cc^{-1}} & 0 \\ 0 & 0 \end{bmatrix} \qquad (2.7)$$

we can express any motion of the mechanism by

$$\dot{x}_k^c = \Sigma_{ki}^o\ \dot{\varepsilon}_i + Z_{kp}^*\ \dot{x}_p^m. \qquad (2.8)$$

We observe that by the particular choice of the basis for the null space determined by (2.6), certain velocity components, $(\dot{x}_p^m)$, are singled out as generalized coordinates of the mechanism with rigid links. In the case of deformable links we have from (2.5) the continuity conditions:

$$Z\ \dot{\varepsilon} = 0, \quad Z = [-B^T \ I]. \qquad (2.9)$$

In (2.6) and (2.7) we have the basis for a kinematic analysis of our four-bar linkage of Fig. 2. The generalization to more complicated mechanisms is easy to visualize.

The second order kinematics of a mechanism is obtained from (2.4) by simple differentiation.

$$\ddot{\varepsilon}_i = D_{i,k}^c\ \ddot{x}_k^c + D_{i,k\ell}^c\ \dot{x}_k^c\ \dot{x}_\ell^c. \qquad (2.10)$$

Here again we have the linear mapping $D_{i,k}$, and now we can write

$$\begin{bmatrix} I & B^* \\ 0 & 0 \end{bmatrix} \ddot{x}^c = \begin{bmatrix} D^{cc^{-1}} & 0 \\ -B^T & I \end{bmatrix} (\ddot{\varepsilon} - D_{,k\ell}^c\ \dot{x}_k^c\ \dot{x}_\ell^c) \qquad (2.11)$$

which leads to the expressions:

$$\ddot{x}^c_k = \Sigma^o_{ki}\,(\ddot{\varepsilon}_i - D^c_{i,\ell m}\,\dot{x}^c_\ell\,\dot{x}^c_m) + Z^*_{kq}\,\ddot{x}^m_q . \tag{2.12}$$

For the determination of the zero'th order kinematics of mechanisms with rigid links ($\varepsilon = 0$, $\dot{\varepsilon} = 0$) the equations (2.8) must be integrated. In a step by step procedure the expressions (2.8) and (2.12) are used as predictors and the conditions $D_i(x^c_k, x^o_\ell) = 0$ as correctors. A detailed account of this development is given in [1]. It formed the basis for the computer programs PLANAR and SPACAR, by which a complete kinematic analysis can be made of almost any mechanism. Even on microcomputers we can deal with fairly complicated mechanisms. We shall discuss later on how in these programs by means of special finite elements sliders, gears and other components can be included.

3. Equations of Motion

Also for the derivation of the equations of motion we start out from the simplest finite element in a mechanism: the connecting rod between two hinges. We note that by taking (2.1) as the sole possible deformation mode, we rule out bending deformations. To include bending it is necessary to introduce in the nodal points the orthogonal transformations for the basevectors, that can represent finite rotations.

At first we shall limit our description to the linear velocity distribution (Fig. 1),

$$\begin{vmatrix} \dot{x}_1 \\ \dot{x}_2 \end{vmatrix} = \begin{bmatrix} \dot{x}^p_1 & \dot{x}^q_1 \\ \dot{x}^p_2 & \dot{x}^q_2 \end{bmatrix} \begin{vmatrix} 1-\xi \\ \xi \end{vmatrix}, \quad \xi = s/\ell, \quad 0 \le \xi \le 1 \tag{3.1}$$

thereby excluding bending deformations.

According to the principle of virtual power, the power of the forces acting on the mechanism must be equal to zero for all velocity distributions that are free of deformation. The nodal forces are elements in the dual space of the space of nodal velocities. In the case of dynamics we have to add the inertial forces to the external forces acting in the nodal points. In order to derive the contribution of the inertial forces, we can use the velocity distribution (3.1) and the acceleration distribution derived therefrom. For one link we find for a mass per unit length m:

$$- \ell \int_0^1 m\, \underline{\dot{x}} \cdot \underline{\ddot{x}}\, d\xi = - \left| \dot{x}^p_1 \; \dot{x}^p_2 \; \dot{x}^q_1 \; \dot{x}^q_2 \right| \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \begin{vmatrix} \ddot{x}^p_1 \\ \ddot{x}^p_2 \\ \ddot{x}^q_1 \\ \ddot{x}^q_2 \end{vmatrix}$$
$$= - \dot{x}^T M \ddot{x} . \tag{3.2}$$

The matrix M is known in the finite element theory as the consistent mass-matrix for the distribution (3.1).

Now we can formulate the principle of virtual power for our four-bar linkage of Fig. 2 by

$$\dot{x}^c_k\,(f^c_k - M_{k\ell}\ddot{x}^c_\ell) = 0 \;\forall\; \dot{x}^c_k \in \{\dot{x}^c_k \mid D^c_{i,k}\,\dot{x}^c_k = 0\}. \tag{3.3}$$

We may include the subsidiary conditions in the equation of virtual power by means of lagrangian multipliers. We then have

$$\dot{x}^c_k\,(f^c_k - M_{k\ell}\,\ddot{x}^c_\ell) = \sigma_i\,D^c_{i,k}\,\dot{x}^c_k \quad \forall\; \dot{x}^c_k \tag{3.4}$$

or

$$\sigma_i\,D^c_{i,k} + M_{k\ell}\,\ddot{x}^c_\ell = f^c_k . \tag{3.5}$$

The lagrangian multipliers are to be interpreted as generalized stresses. It is easily seen that in the static case ($\ddot{x}^c_\ell = 0$) for links of the type of Fig. 1 they represent the normal forces in the trussmembers.

Again we meet a linear mapping; this time represented by a matrix, which is the transposed matrix of the mapping (2.4). By similar transformations as applied to (2.4) in order to arrive at (2.5), we now find

$$\begin{bmatrix} I & B \\ 0 & 0 \end{bmatrix} \sigma = \begin{bmatrix} [D^{ccT}]^{-1} & 0 \\ -B^{*T} & I \end{bmatrix} (f^c - M\ddot{x}^c) \tag{3.6}$$

or

$$\sigma_i = \Sigma^o_{ki}\,(f^c_k - M_{k\ell}\,\ddot{x}^c_\ell) + Z_{pi}\,\sigma^r_p , \tag{3.7}$$

$$Z^*_{kp}\,(f^c_k - M_{k\ell}\,\ddot{x}^c_\ell) = 0. \tag{3.8}$$

In the case of undeformable links ($\varepsilon = 0$) with a statically determinate structure ($Z_{pi} = 0$) by substitution of (2.8) and (2.12) into (3.8) we can reduce the equations of motion to a system of differential equations for the generalized coordinates x^m.

$$Z^{*T} M Z^* \,\ddot{x}^m = Z^{*T}(f^c - M\Sigma^o g), \tag{3.9}$$

with

$$g_i = D^c_{i,mn}\,\dot{x}^c_m\,\dot{x}^c_n , \qquad \dot{x}^c_m = Z^*_{mp}\,\dot{x}^m_p . \tag{3.10}$$

It should be realized however that the elements of the matrices Z^* and Σ^o depend nonlinearly on the nodalcoordinates, while for more general elements this is also the case for $D_{i,mn}$ and the inertial terms. Hence a numerical integration of the system (3.9) must be accompanied by a simultaneous determination of the zero'th order kinematics, such as it was indicated in the previous section. Because of the statical determinacy the generalized stresses σ_i can be calculated after any step in the integration process. The initial conditions for x^m_p and $\dot{x}^m_p$ follow from the initial conditions for the nodal coordinates as soon as the analysis of the linear mapping (2.4) has furnished the coordinates, which are singled out as the generalized coordinates x^m_p.

If we have to take into account one or more deformation modes in the structure the equations of motion and the equations for the generalized stresses are directly coupled by these deformation modes, unless the deformations are prescribed as functions of time. It will be clear that in the latter case we are not dealing with simple elements such as the connecting rod of Fig. 1. As we shall see in section 4 the role of actuators in mechanisms can be described by means of elements with prescribed deformations.

If the deformations depend on the generalized stresses, then the equations (3.7) and (3.8) have to be treated as a coupled system. Again by substitution of the expressions (2.8) and (2.12) we arrive at the reduced system of differential equations; this time in terms of the systemparameters x^m and ε

$$\begin{bmatrix} Z^{*T}MZ^{*} & Z^{*}M\Sigma^{o} \\ \Sigma^{oT}MZ^{*} & \Sigma^{oT}M\Sigma^{o} \end{bmatrix} \begin{vmatrix} \ddot{x}^m \\ \ddot{\varepsilon} \end{vmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & S \end{bmatrix} \begin{vmatrix} x^m \\ \varepsilon \end{vmatrix} =$$

$$= \begin{vmatrix} Z^{xT} \\ \Sigma^{oT} \end{vmatrix} (f^c - M\Sigma^o g) - \begin{vmatrix} 0 \\ Z^T S^r D^r \end{vmatrix}, \qquad (3.11)$$

with

$$g_i = D^c_{i,mn}\, \dot{x}^c_m\, \dot{x}^c_n\, , \qquad \dot{x}^c_m = \Sigma^o_{mi}\, \dot{\varepsilon}_i + Z^*_{mp}\, \dot{x}^m_p\, . \qquad (3.12)$$

The analysis of the linear mapping (2.4) provides the generalized coordinates x^m_p, while the dual linear mapping, occurring in (3.5), determines whether there are any statically indeterminate parts in the structure. If this is the case, then the transformed matrix in (3.6) singles out the stresses, which may be denoted as the redundant stresses σ^r. For elastic links these redundant stresses are uniquely determined by the corresponding deformations $D^r(x^c_k, x^o_\ell)$ and for purely viscous links by the rate of the corresponding deformations. For visco-elastic behaviour or for more general constitutive equations the momentary value of the redundant stresses is history dependent.

The computerprogram DYNAMO [2] is based upon the system (3.11) and consequently it can deal with mechanisms with elastic links as well as with links with prescribed deformations. Essential for the program is the incorporation of a numerical integration routine, that can automatically adjust the time step and the order of integration. As is wellknown high natural frequencies, resulting from elastic, but very stiff links require a relatively small time step.

4. Illustrative Examples

It is not possible within the context of this paper to discuss in any detail the various finite elements, that have been devised to represent the mechanical behaviour of components of mechanisms. Fig. 3 gives a picture of elements, that are used in the following examples.

For the numerical determination of the angular orientation we found Euler parameters most suitable. As soon as the angular orientation has to be taken into account the configuration can no longer be determined solely by vectors, because the rotations can only be characterized by orthogonal transformations of the base vectors. A vector with direction cosines $\cos\alpha$, $\cos\beta$, $\cos\gamma$ and an angle of rotation μ about this vector determines the Euler parameters by

$$\lambda_0 = \cos\tfrac{1}{2}\mu, \quad \lambda_1 = \cos\alpha\sin\tfrac{1}{2}\mu, \quad \lambda_2 = \cos\beta\sin\tfrac{1}{2}\mu, \quad \lambda_3 = \cos\gamma\sin\tfrac{1}{2}\mu, \qquad (4.1)$$

The orthogonal transformation is quadratic in these parameters.

$$R = \begin{bmatrix} \lambda_0^2 + \lambda_1^2 + \lambda_2^2 - \lambda_3^2 & 2(\lambda_1\lambda_2 - \lambda_0\lambda_3) & 2(\lambda_1\lambda_3 + \lambda_0\lambda_2) \\ 2(\lambda_1\lambda_2 + \lambda_0\lambda_3) & \lambda_0^2 - \lambda_1^2 + \lambda_2^2 - \lambda_3^2 & 2(\lambda_2\lambda_3 - \lambda_0\lambda_1) \\ 2(\lambda_1\lambda_3 - \lambda_0\lambda_2 & 2(\lambda_2\lambda_3 + \lambda_0\lambda_1) & \lambda_0^2 - \lambda_1^2 - \lambda_2^2 + \lambda_3^2 \end{bmatrix}. \qquad (4.2)$$

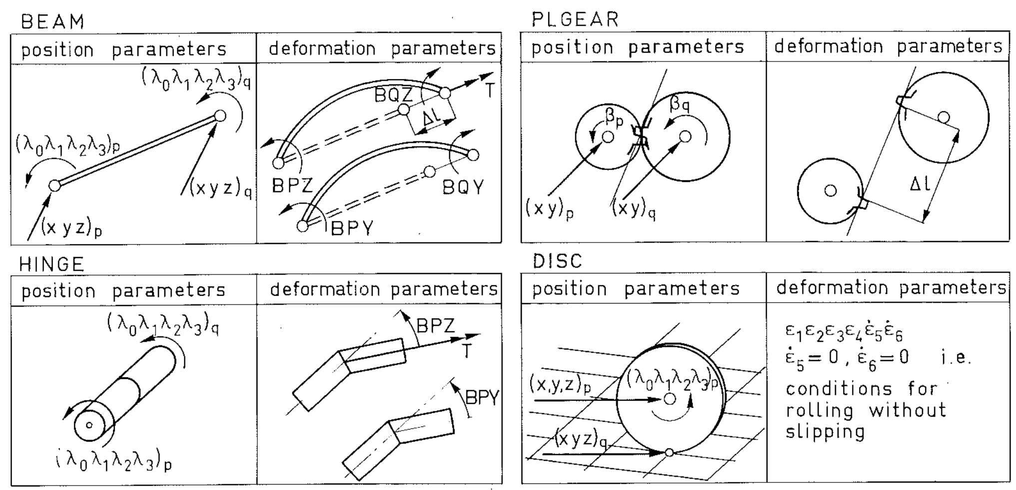


Fig. 3. Finite element representations of mechanism components
(ΔL = elongation, B = bending, T = torsion)

The transformation matrices for expressing time derivatives of Euler parameters in terms of vector components of angular velocities and accelerations are linear in the parameters and require therefore little time of computation.

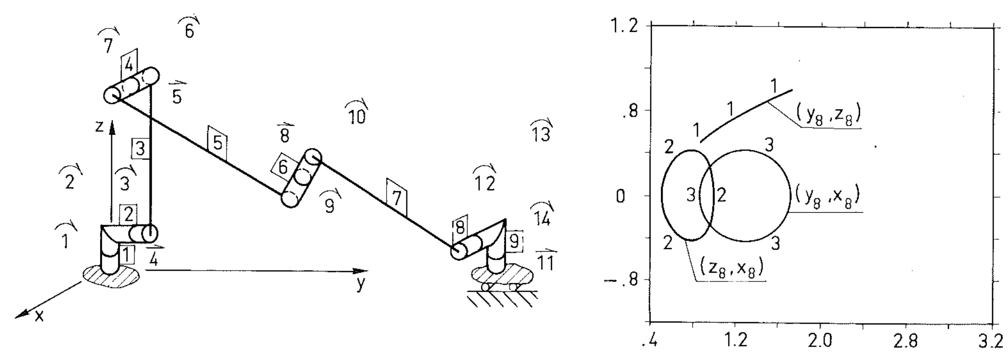
The subsidiary condition for the Euler parameters,

$$\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 1 = 0 \qquad (4.3)$$

is of similar form as an undeformability condition ($\varepsilon = 0$) and is therefore introduced in the description under the name λ-element.

As it was already mentioned the inertial terms are generally more complicated than the expressions given in (3.2). Only if the velocity distribution is represented by isotropic polynomials a massmatrix, independent of orientation, is obtained. Otherwise there is a dependence on orientation, which can be expressed in terms of Euler parameters. Rotating subsystems, forming gyrostats, may be represented as lumped rotational inertias having particular dynamic properties. Details are given in [2] and [3].

Our first example is the 3D, so-called turbula mechanism of Fig. 4. Here we meet the beamelement and the hinge-element, both requiring the description of arbitrarily large rotations. The orthogonal transformations in the expressions for the deformation parameters, in [4] defined in terms of Euler angles, were for mechanisms expressed in terms of Euler parameters in order to avoid the singularity orientations, which occur in the description by Euler angles. The system parameters ε, besides bending of beams, now also represent relative rotations in hinges. These can be split up in prescribed system parameters, ε^m, and calculable parameters, ε^c. The prescribed rotation, ε_T^1, gives rise to relative rotations in the other five hinges and determines the path of coordinate 8, as pictured in Fig. 4.



Element 1, 2, 4, 6, 8, 9: hinge
Element 3, 5, 7 : beam

3 projections of the path of coordinate 8.

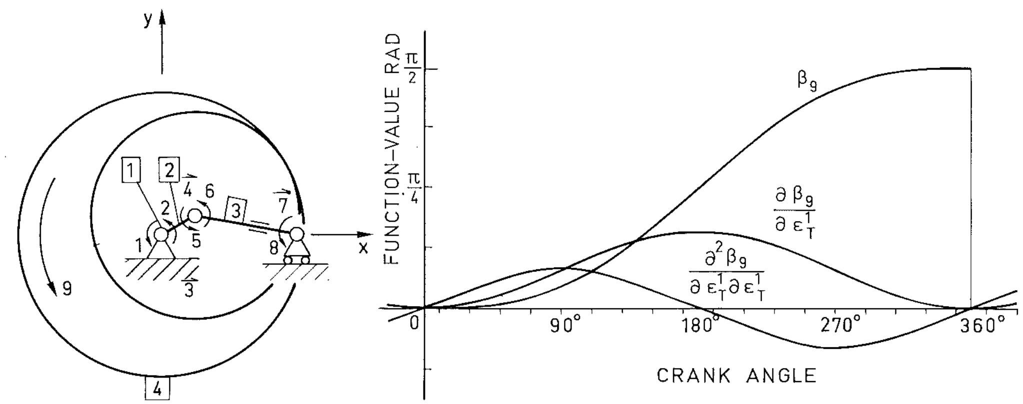
$x^m = \{\emptyset\}$ $\quad x^o = \{\vec{x}_4, x_{11}, z_{11}, \vec{\lambda}_1, \vec{\lambda}_{11}\}$

$\varepsilon^m = \{\varepsilon_T^1\}$ $\quad \varepsilon^c = \{\varepsilon_T^2, \varepsilon_T^4, \varepsilon_T^6, \varepsilon_T^8, \varepsilon_T^9\}$

Fig. 4. Kinematic analysis of 3D turbula mechanism [5].

Since gears are important components of mechanisms their kinematical behaviour is also described by finite elements. After early versions, discussed

in [6], the third author devised an element, that more explicitely represents the characteristics of a gear pair. Details will be published soon. Fig. 5 shows as an example the kinematic analysis of a 2D gear-bar mechanism with four rests.



element 4: plgear

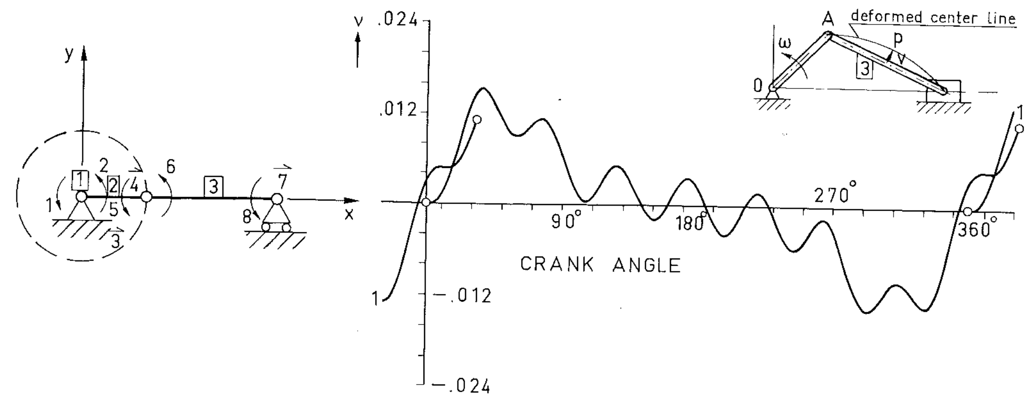
$x^m = \{\emptyset\}$ $\quad x^o = \{\beta_1, \vec{x}_3, y_7\}$

$\varepsilon^m = \{\varepsilon_T^1\}$, $\varepsilon^c = \{\emptyset\}$

0, 1 and 2-order transfer function of β_9 as function of ε_T^1.

Fig. 5. Kinematic analysis of 2D gear-bar mechanism with 4 rests.

Our third example deals with the dynamic behaviour of a 2D crank-slider mechanism. The connecting rod is elastic in bending. The other links are rigid. The results could be compared with results in the literature, obtained as a result of an ad hoc analysis for this particular case. Computed with the program DYNAMO our results are just an illustration of the capabilities of this program (Fig. 6).



Element 1 : pltor

Element 2, 3: plbeam

$x^m = \{\emptyset\}$ $x^o = \{\beta_1, \bar{x}_3, y_7\}$

$\varepsilon^m = \{\varepsilon^1_T\}$ $\varepsilon^c = \{\emptyset\}$ $\varepsilon^d = \{\varepsilon^3_{BPZ}, \varepsilon^3_{BQZ}\}$

Displacement of the midpoint of element 3 with respect to the undeformed center line, ie. v.

Fig. 6. Dynamic analysis of 2D crank-slider mechanism with elastic connecting rod.

As it was already mentioned rotating subsystems can be included in the dynamical analysis as so-called gyrostats; lumped rotational inertias having particular dynamic properties. The gyroscope shown in Fig. 7 consists of a uniform thin disk spinning freely about the end of a rigid massless rod. The other end is supported at a fixed point 0. The initial conditions are:

$$\theta(0) = 0, \quad \dot{\theta}(0) = 0 \quad \psi(0) = 0, \ \dot{\psi}(0) = \frac{2a^2}{a^2+4\ell^2}\,|\vec{\Omega}|,$$

where $\dot{\psi}/0$ is the initial precession rate and $\vec{\Omega}$ is the total spin vector. For these conditions the rod moves through the bottom position at $\theta = -\frac{\pi}{2}$.

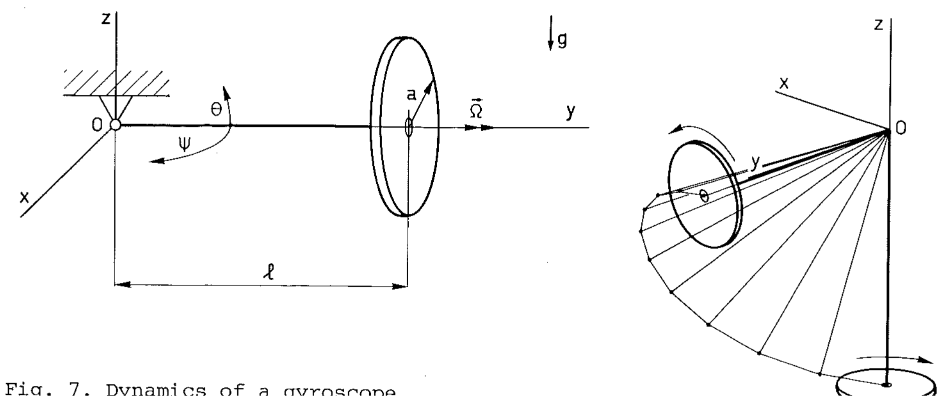
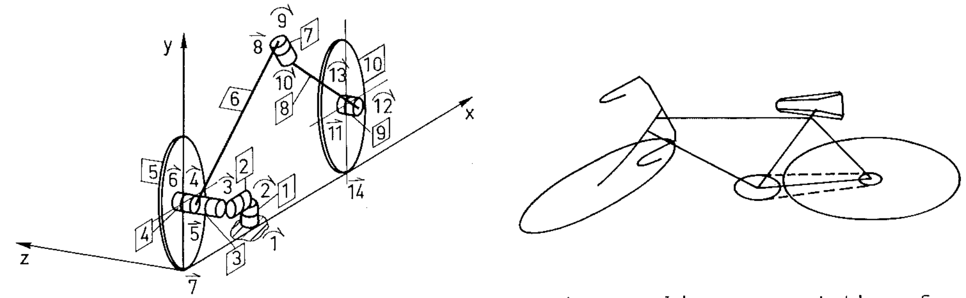


Fig. 7. Dynamics of a gyroscope

We have room for one more example: the kinematic analysis of a 3D bycicle. For a wheel on a plane four deformation parameters can be defined. Sliding can only be looked upon as two non-integrable rates of deformation. Since it is kinematics we are dealing with, we have to prescribe three rotations for a motion of the bycicle. The model is shown in Fig. 8.



Computer graphic representation of the analysis.

Element 1, 2, 3, 4, 7, 9: hinge

Element 6, 8 : beam

Element 5, 10 : disc

$x^m = \{\emptyset\}$ $x^o = \{\bar{\lambda}_1\}$

$\varepsilon^m = \{\varepsilon^4_T, \varepsilon^7_T, \varepsilon^2_T\}$ $\varepsilon^c = \{\varepsilon^1_T, \varepsilon^3_T, \varepsilon^9_T\}$

Fig. 8. Kinematic analysis of 3D bycicle.

If the analysis is extended to dynamic behaviour by equations of the type (3.11), then a control system has to be added in order to direct the bycicle along a given path. In fact the combination of our present tools for dynamic analysis of mechanisms with the tools of control theory is a challenge for the coming years. In particular this combination will be studied in connection with work in the field of robotics.

References

1. K. van der Werff, Kinematic and Dynamic Analysis of Mechanisms, A Finite Element Approach, *Dissertation T.H.-Delft* (Delft University Press, 1977).

2. A.L. Schwab, Dynamica van Mechanismen met Vervormbare Schakels, *Engineer's Thesis* (Lab. of Eng. Mech., Rapport nr. 758, 1983).

3. J.B. Jonker, Dynamics of Spatial Mechanisms with Flexible Links. To be published as WTHD-report no. 171 (Lab. of Eng. Mech., Dept. Mech. Eng., 1985).

4. J.F. Besseling, Derivatives of Deformation Parameters for Bar Elements and their use in Buckling and Post-buckling Analysis, *Comp. Meths. Appl. Mech. Eng.* 12 (1977) pp. 97-124.

5. M.S. Konstantinov and M.D. Markov, Discrete Positions Method in Kinematic and Control of Spatial Mechanisms, *Mechanism and Machine Theory*, 15 (1980) pp. 47-60.

6. J.F. Besseling, L.J. Ernst, K. van der Werff, A.U. de Koning and E. Riks, Geometrical and Physical Nonlinearities, Some Developments in the Netherlands, *Comp. Meths. Appl. Mech. Eng.* 17/18 (1979) pp. 131-157.

7. SPACAR User Manual, *Publication of the CADOM group* (Delft Univ. of Techn., Dept. of Mech. Eng., 1984).

Parametric Resonance and Instability in Offshore Structures

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Motivated by questions concerning the possibility of parametrically excited resonance and instability of certain types of marine structures, analysis has been made of the non-stationary response of a homogeneous non-linearly damped second-order system with randomly fluctuating restoring coefficient and subjected to some initial disturbance at time zero. The analysis has resulted in some new solutions which are discussed in this paper. paper.

1. Introduction

There is considerable interest in the phenomenon of unbounded solutions of second-order differential equations with restoring force coefficients which vary in magnitude with time. Part of this interest comes from ocean engineering circles and concerns the possible unstable motion of certain types of marine structures. Examples are roll motion of ships or, more recently, horizontal motion of compliance structures for oil and gas production.

Conclusions with regard to unstable motion often refer to unbounded solutions of the Mathieu equation. Results thus obtained, however, are only valid for second-order systems with sinusoidally varying coefficients. In marine applications, rather than sinusoidally, restoring coefficients vary randomly in magnitude with time. Some recent results for this new type of problem are discussed. Effects of non-linear damping are considered and implications for marine structures are indicated.

A summary of the adopted solution procedure is presented in Figure 1.