5th Asian Conference on Multibody Dynamics 2010
August 23–26, 2010, Kyoto, Japan

Controllability of a bicycle

A. L. Schwab
Laboratory for Engineering Mechanics
Delft University of Technology
Mekelweg 2, NL-2628 CD Delft, The Netherlands
email: a.l.schwab@wbmt.tudelft.nl

J. D. G. Kooijman
Laboratory for Engineering Mechanics
Delft University of Technology
Mekelweg 2, NL-2628 CD Delft, The Netherlands
email: jodikooijman@gmail.com

Keywords: Bicycle dynamics, human control, nonholonomic systems, multibody dynamics.

Abstract

The bicycle is an intriguing machine as it is laterally unstable at low speed and stable, or easy to stabilize, at high speed. During the last decade a revival in the research on dynamics and control of bicycles has taken place [1]. Most studies use the so-called Whipple [2] model of a bicycle. In this model a rigid rider is rigidly connected to the rear frame. However, from experience it is known that some form of control is required to stabilize the bicycle and/or carry out tracking operations. This control is either carried out by steering or by performing some sort of upper body motions. Note that in both cases the system is underactuated. The precise control used by the rider is still under study [3]. This paper addresses the question whether the underactuated bicycle is controllable by only steering or upper body motion in the forward speed range of 0 to 36 km/h. Whipple-like models are studied with either steering or upper body control. It is shown that at certain specific forward speeds some modes are uncontrollable. However, either the forward speed is extremely low or the uncontrollable modes are all stable modes and are therefore of no concern to the rider.

1 Introduction

The bicycle is an intriguing machine as it is laterally unstable at low speed and stable, or easy to stabilize, at high speed. During the last decade a revival in the research on dynamics and control of bicycles has taken place [1]. Most studies use the so-called Whipple model [2] of a bicycle. In this model a hands-free rigid rider is fixed to the rear frame. However, from experience it is known that some form of control is required to stabilize the bicycle and/or carry out tracking operations. This control is either done by steering or by performing some set of upper body motions. The precise control used by the rider is currently under study [4; 3].

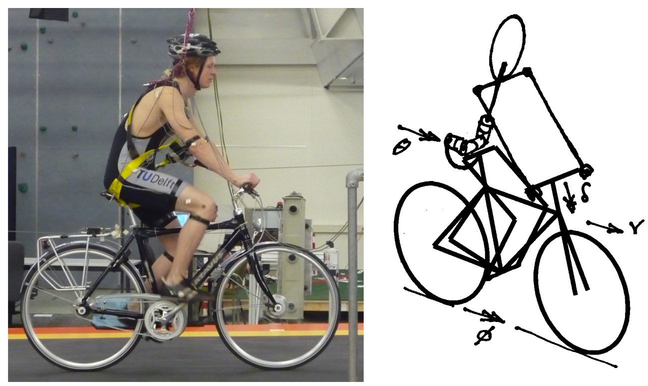

Figure 1. Rider posture on the Stratos bicycle with a forward leaned body and stretched arms on the handle bars. The degrees of freedom are the rear frame lean angle ϕ, the upper body lean angle θ, the steer angle δ, and the forward speed v. The upper body pitch and twist are passive motions, in the sense that they do not add any degrees of freedom to the system.

Here we focus on the controllability of the system. We consider two inputs: steer torque or upper body lean torque. For the model we use an extended Whipple model where the upper body with arms, connected to the handle bars, has been added in two different ways, guided by observations [4; 3]. In the first posture, Fig 1, the rider leans forward and holds the handle bars with stretched arms. In the second posture, Fig 2, the rider sits straight up, with flexed arms connected to the handle bars. Apart from the lateral lean motion of the upper body, these models do not add any degrees of freedom to the original Whipple model [5].

Among the few people who have investigated controllability of a bicycle, Nagai [6] used a Whipple like bicycle model with steer angle and upper body angle input control. He finds one non-zero forward speed and one mass distribu-

Copyright © 2010 by JSME

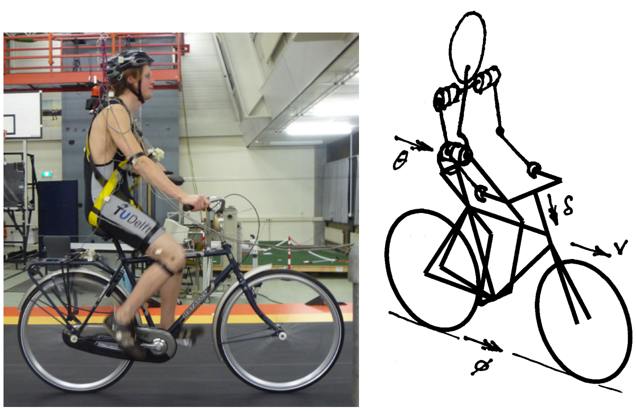

Figure 2. Rider posture on the Browser bicycle with an upright body and flexed arms on the handle bars. The degrees of freedom are the rear frame lean angle ϕ, the upper body lean angle θ, the steer angle δ, and the forward speed v. The motion of the arms, connected to the handle bars, are passive, in the sense that they do not add any degrees of freedom to the system.

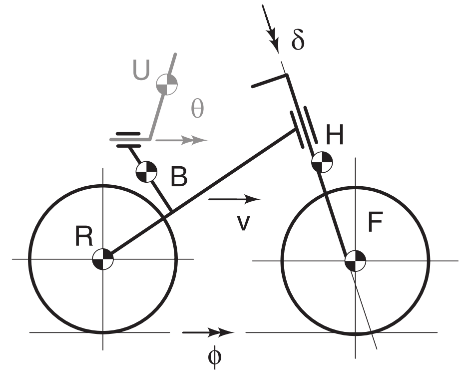

Figure 3. The extended bicycle model: five rigid bodies (rear wheel R, rear frame B, upper body U, front handlebar assembly H, front wheel F) connected by four revolute joints (rear hub, saddle, steering axis, front hub), together with the coordinate system, and the degrees of freedom: forward speed v, rear frame lean angle ϕ, steer angle δ, and upper body lean angle θ. Note that the arms are connected in two different ways to the steering assembly, see Fig 1 and 2.

tion which result in an uncontrollability for the system. Seffen *et al.* [7] investigate uncontrollability for a Whipple like bicycle model and introduces an index which should indicate the difficulty of riding. The index is based on the ratio of the singular values of the controllability matrix (7). Neither work addresses whether the uncontrollable mode is stable or unstable. It could well be that the uncontrollable or near to uncontrollable mode is a stable mode of the system and therefore of no concern to the rider. This paper tries to resolve that problem by determining the forward speed at which the bicycle is uncontrollable and then identify if this is a stable or unstable mode. It also introduces a transfer function for modal control from which controllability can be determined in a continuous way.

The paper is organized as follows. First the bicycle plus rider model is presented. Next the controllability is investigated for two types of posture, bend forward with straight arms and straight up with flexed arms. The paper ends with some conclusions.

2 Bicycle plus Rider Model

The basic bicycle model used is the so-called Whipple [2] model which recently has been benchmarked [1]. This model is then extended with an upper body which is able to lean laterally. This basic model, see Figure 3, consists of five rigid bodies connected by revolute joints. The arms are then connected to the handle bars in two different manners, stretched or flexed, see Fig. 1 and 2, where necessary hinges are added to the system in such a manner that no degrees of freedom are added to the system. The contact between the knife-edge wheels and the flat level surface is modelled by holonomic constraints in the normal direction and by non-holonomic constraints in the longitudinal and lateral direction. The resulting non-holonomic mechanical model has four velocity degrees of freedom: forward speed v, rear frame lean rate $\dot{\phi}$, steer rate $\dot{\delta}$, and the upper body lean rate $\dot{\theta}$.

For the stability and controllability analysis of the lateral motions we consider the linearized equations of motion for small perturbations about the upright steady forward motion. They are expressed in terms of small changes in the lateral degrees of freedom (the rear frame roll angle, ϕ, the steer angle, δ, and the upper body lean angle θ) from the upright straight ahead configuration $(\phi,\delta,\theta) = (0,0,0)$, at a forward speed v, and have the form

$$\mathbf{M}\ddot{\mathbf{q}} + v\mathbf{C_1}\dot{\mathbf{q}} + [g\mathbf{K_0} + v^2\mathbf{K_2}]\mathbf{q} = \mathbf{f}, \tag{1}$$

where the time-varying variables are $\mathbf{q} = [\phi,\delta,\theta]^T$ and the lean and steering torques $\mathbf{f} = [T_\phi, T_\delta, T_\theta]^T$. The coefficients in this equation are: a constant symmetric mass matrix, $\mathbf{M}$, a damping-like (there is no real damping) matrix, $v\mathbf{C_1}$, which is linear in the forward speed v, and a stiffness matrix which is the sum of a constant symmetric part, $g\mathbf{K_0}$, and a part, $v^2\mathbf{K_2}$, which is quadratic in the forward speed. The forces on the right-hand side, $\mathbf{f}$, are the applied forces which are energetically dual to the degrees of freedom $\mathbf{q}$.

The complete model of the bicycle with passive rider was analyzed with the multibody dynamics software package SPACAR [8]. SPACAR handles systems of rigid and flexible bodies connected by various joints in both open and closed kinematic loops, and where parts may have rolling contact. SPACAR generates numerically, and solves, full non-linear dynamics equations using minimal coordinates (constraints

Copyright © 2010 by JSME

are eliminated). SPACAR can also find the numeric coefficients for the linearized equations of motion based on a semi-analytic linearization of the non-linear equations.

For the modelling of the geometry and mass properties of the rider, the method as described by Moore *et al.* 2009 [9] is used. Here the human rider is divided into a number of simple geometric objects like cylinders, blocks and a sphere of constant density. Then with the proper dimensions and the estimates of the individual body part masses the mechanical models can be made. The geometry and mass properties of the two bicycles and the rider used in this study where measured by the procedure as described in [9] and the results are presented in [5].

Then, with the coefficient matrices the characteristic equation,

$$\det\left(\mathbf{M}\lambda^2 + v\mathbf{C_1}\lambda + g\mathbf{K_0} + v^2\mathbf{K_2}\right) = 0, \tag{2}$$

can be formed and the eigenvalues, λ, can be calculated. These eigenvalues, in the forward speed range of $0 \leq v \leq 10$ m/s, are presented, for example for the Stratos bicycle with forward leaned rider with stretched arms on the handle bars, in Fig. 4a. In principle there are up to six eigenmodes, where oscillatory eigenmodes come in pairs. Two are significant and are traditionally called the *capsize mode* and *weave mode*. The capsize mode corresponds to a real eigenvalue with eigenvector dominated by rear frame lean: when unstable, the bicycle just falls over like a capsizing ship. The weave mode is an oscillatory motion in which the bicycle sways about the headed direction. The third eigenmode is the overall stable *castering mode*, like in a caster wheel, which corresponds to a large negative real eigenvalue with eigenvector dominated by steering. The remaining pair of eigenmodes are the upper body lean motion relative to the rear frame, where the positive root in this pair corresponds to falling, whereas the negative root corresponds to the time reversal of this falling. This pair usually shows little dependency in the forward speed. The upper body lean mode is always unstable but will be stabilized in the usual manor like in sitting upright at rest.

3 Controllability

To investigate the controllability of the bicycle rider system we rewrite the linearized equations of motion (1) into a set of first order differential equations, the so-called state-space form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \tag{3}$$

with the state vector $\mathbf{x}$ and the control input vector $\mathbf{u}$. For the bicycle the state vector is $\mathbf{x} = [\phi, \delta, \theta, \dot{\phi}, \dot{\delta}, \dot{\theta}]^T$ and the control input vector is $\mathbf{u} = [T_\delta, T_\theta]^T$. Since we wish to address the control inputs separately, we split the input vector $\mathbf{u}$ and the associated matrix $\mathbf{B}$ into two scalars and two associate vectors,

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}_\delta T_\delta + \mathbf{B}_\theta T_\theta. \tag{4}$$

For the bicycle rider system the coefficient matrix $\mathbf{A}$ and vectors $\mathbf{B}_\delta$ and $\mathbf{B}_\theta$ are given by

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}(g\mathbf{K_0} + v^2\mathbf{K_2}) & -\mathbf{M}^{-1}(v\mathbf{C_1}) \end{bmatrix}, \tag{5}$$

$$\mathbf{B}_\delta = \begin{bmatrix} \mathbf{0} \\ \mathbf{M}^{-1}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \end{bmatrix}, \quad \mathbf{B}_\theta = \begin{bmatrix} \mathbf{0} \\ \mathbf{M}^{-1}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{bmatrix}. \tag{6}$$

3.1 Standard approach

In the standard approach to determine controllably of a linear dynamical system like (4), the controllability matrix

$$\mathbf{Q}_j = [\mathbf{B}_j, \mathbf{A}\mathbf{B}_j, \mathbf{A}^2\mathbf{B}_j, \cdots, \mathbf{A}^{k-1}\mathbf{B}_j], \tag{7}$$

is formed and if this controllability matrix has full rank k, where k is the order of the system and equal to the number of states, then the system is fully controllable by input $j = (\delta, \theta)$. Here, we investigate rank deficiency by setting the determinant of $\mathbf{Q}_j(v)$ to zero. With the forward speed v as a parameter, this leads to a characteristic equation in v. The solutions are the forward speed for which the system is uncontrollable, which we call v_u. The corresponding eigenvector, $\mathbf{v}_u$, is the null space of the transpose of the corresponding controllability matrix, $\mathbf{v}_u = \text{null}(\mathbf{Q}_j^T(v_u))$. Since this is also an eigenvector of the system matrix $\mathbf{A}(v_u)$, the corresponding eigenvalue λ_u can be found from the definition $\mathbf{A}\mathbf{v}_u = \lambda_u\mathbf{v}_u$. This procedure has been applied to both bicycle plus rider models and the results are presented in Tables 1,2 and Fig. 4,5.

For the Stratos bicycle with the forward leaned rider with stretched arms on the handle bars, controlled by steer torque control, Table 1 Fig. 4a, we find four uncontrollable forward speeds. However, only the one at 1.8174 m/s concerns an unstable mode, but since this is an upper body lean mode it is of no concern to the controllability of the bicycle. If we consider only upper body lean torque control then there are two uncontrollable forward speeds, but again only one, now at 0.0084 m/s, concerns an unstable mode. This mode is the prequel to the oscillatory weave mode but since the speed is almost zero this is again of no concern to the practical control of the bicycle. We conclude that this bicycle rider configuration is fully controllable by either steer torque control or upper body torque control.

Copyright © 2010 by JSME

Steer torque control, T_δ

v_u [m/s]	λ_u [rad/s]	$(\phi,\delta,\theta)_u$ [rad]	mode
0.0049	-3.0150	$(0.13,\ 0.59, -0.80)$	capsize
1.5477	-3.0150	$(0.15,\ 0.72, -0.68)$	capsize
1.8174	7.8252	$(0.15,\ 0.75, -0.65)$	lean
4.5533	-7.8252	$(0.06, -0.28,\ 0.96)$	lean

Upper body lean torque control, T_θ

v_u [m/s]	λ_u [rad/s]	$(\phi,\delta,\theta)_u$ [rad]	mode
0.0084	3.0164	$(0.13,\ 0.59, -0.80)$	w1
1.5004	-3.0219	$(0.15,\ 0.72, -0.68)$	capsize

Table 1. Forward speed v_u at which the Stratos bicycle with the forward leaned rider with stretched arms on the handle bars from Fig. 1, is uncontrollable by either steer torque control T_δ or upper body lean torque control T_θ together with the corresponding eigenvalue λ_u and eigenvector coordinates $(\phi,\delta,\theta)_u$, with rear frame lean angle ϕ, steer angle δ and upper body lean angle θ, together with the mode description, see also Fig. 4

Steer torque control, T_δ

v_u [m/s]	λ_u [rad/s]	$(\phi,\delta,\theta)_u$ [rad]	mode
0.0124	-2.8979	$(0.16,\ 0.48, -0.86)$	capsize
0.8476	6.5908	$(0.14,\ 0.44, -0.89)$	lean
1.0119	-2.8979	$(0.14,\ 0.44, -0.89)$	capsize
4.0951	-6.5908	$(0.01, -0.26,\ 0.97)$	lean2

Upper body lean torque control, T_θ

v_u [m/s]	λ_u [rad/s]	$(\phi,\delta,\theta)_u$ [rad]	mode
0.2722	2.9013	$(0.15, 0.47, -0.87)$	capsize
1.2336	-2.9120	$(0.13, 0.42, -0.90)$	caster

Table 2. As Table 1 but now for the Browser bicycle with an upright rider and flexed arms on the handle bars from Fig. 2, see also Fig. 5

For the Browser bicycle with an upright rider and flexed arms on the handle bars we first identify that the eigenvalue structure differs vastly from the Stratos bicycle with rider configuration. Whereas the Stratos had a stable forward speed range, between 6.9 and 8.7 m/s, the Browser configuration is always unstable. Although the weave mode is always stable, there is now a capsize mode which is always unstable. For steer torque control on the Browser configuration, Table 2 and Fig. 5a, we find again four uncontrollable forward speeds, where only the one at 0.8476 concerns an unstable mode. This is again an upper body lean mode and therefore of no concern to the stability of the bicycle. For upper body lean control we have two controllable speeds, where only the one at 0.2722 m/s concerns an unstable capsize mode. But since this is also at a very low speed, one can say that, from a

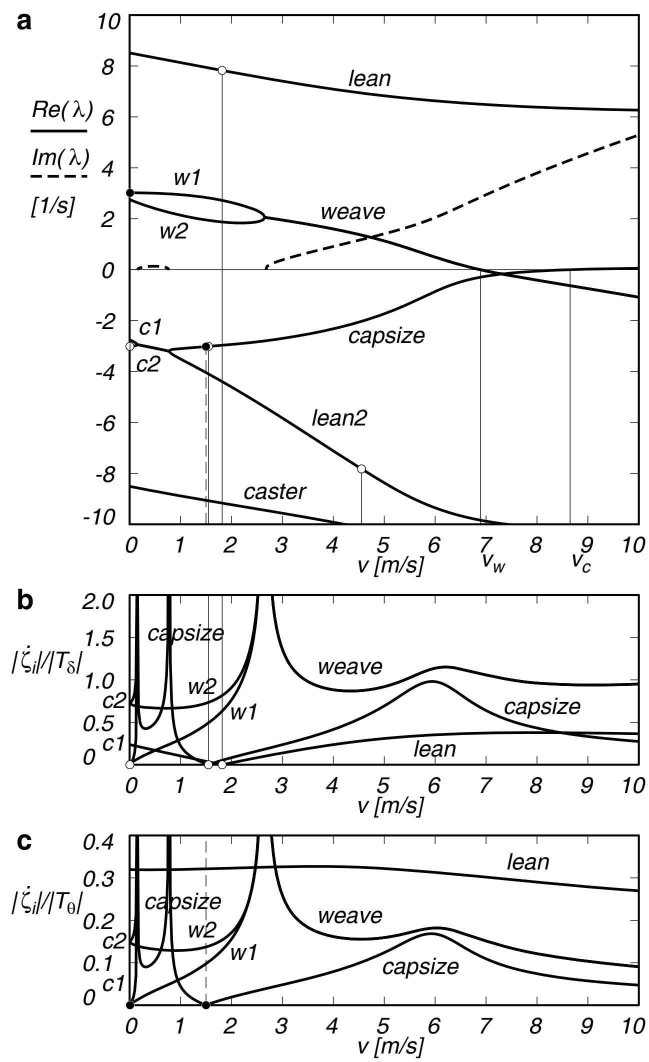

Figure 4. **a**) Eigenvalues λ from the linearized stability analysis for the Stratos bicycle with the forward leaned rider with stretched arms on the handle bars from Fig. 1, where the solid lines correspond to the real part of the eigenvalues and the dashed line corresponds to the imaginary part of the eigenvalues, in the forward speed range of $0 < v < 10$ m/s, together with forward speeds for which the bicycle is uncontrollable by either steer torque (○—) or upper body lean torque (● - -). **b**) Magnitude of the transfer function of the eigenmode rate $\dot\zeta_i$ to the steer torque control torque T_δ for this bicycle model. **c**) Magnitude of the transfer function of the eigenmode rate $\dot\zeta_i$ to the upper body control lean torque T_θ for this bicycle model.

Copyright © 2010 by JSME

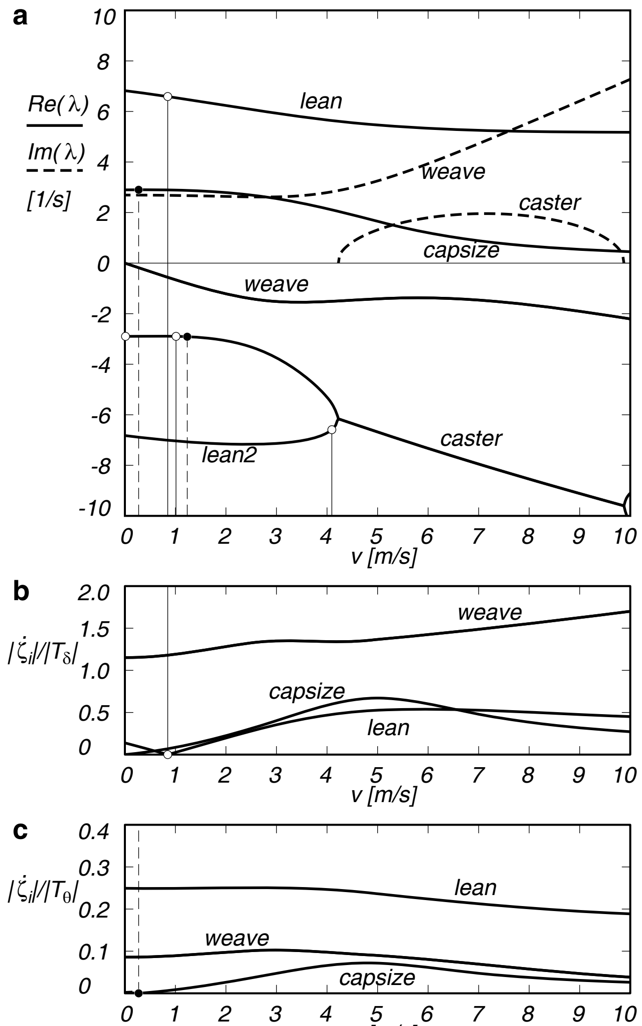

Figure 5. **a**) Eigenvalues λ from the linearized stability analysis for the Browser bicycle with an upright rider and flexed arms on the handle bars from Fig. 2, where the solid lines correspond to the real part of the eigenvalues and the dashed line corresponds to the imaginary part of the eigenvalues, in the forward speed range of $0 < v < 10$ m/s, together with forward speeds for which the bicycle is uncontrollable by either steer torque (○—) or upper body lean torque (● - -). **b**) Magnitude of the transfer function of the eigenmode rate $\dot{\zeta}_i$ to the steer torque control torque T_δ for this bicycle model. **c**) Magnitude of the transfer function of the eigenmode rate $\dot{\zeta}_i$ to the upper body control lean torque T_θ for this bicycle model.

practical point of view, this configuration is also fully controllable by either steer torque control or upper body lean torque.

3.2 Modal controllability

The approach from above, results in a discrete set of velocities for which the bicycle is uncontrollable. It does not tell us anything about the ease or difficulty by which the bicycle is to control in the neighborhood of these uncontrollable speeds. To investigate that, we follow a somewhat different approach and look at the modal controllability.

We start with the system matrix $\mathbf{A}$ for which we determine all eigenvectors $\mathbf{v}_i$ from $\mathbf{AV} = \mathbf{V\Lambda}$, with the modal matrix $\mathbf{V}$ where its columns are the eigenvectors $\mathbf{v}_i$ and the diagonal matrix $\mathbf{\Lambda}$ with the eigenvalues λ_i on the diagonal. If the eigenvalues are distinct and the eigenvectors span the complete state space we can transform the state equations (4) to their modal form by substitution of $\mathbf{x} = \mathbf{V}\boldsymbol{\zeta}$ and premultiplying by the inverse of the modal matrix $\mathbf{V}^{-1}$, resulting in the modal state equations,

$$\dot{\boldsymbol{\zeta}} = \mathbf{\Lambda}\boldsymbol{\zeta} + \bar{\mathbf{B}}_\delta T_\delta + \bar{\mathbf{B}}_\theta T_\theta, \tag{8}$$

with the modal coordinates $\boldsymbol{\zeta}$ and the modal control input vectors,

$$\bar{\mathbf{B}}_\delta = \mathbf{V}^{-1}\mathbf{B}_\delta, \quad \bar{\mathbf{B}}_\theta = \mathbf{V}^{-1}\mathbf{B}_\theta. \tag{9}$$

The big advantage of the modal transformation is that these modal state equations are now decoupled,

$$\dot{\zeta}_i = \lambda_i\zeta_i + \bar{B}_{i\delta}T_\delta + \bar{B}_{i\theta}T_\theta, \quad i = 1\cdots k, \tag{10}$$

and that the controllability follows directly from zero entries in the control matrix $\bar{\mathbf{B}}$. If the entry $\bar{B}_{ij}$ is zero then eigenmode i is uncontrollable by input j. Moreover, around the uncontrollability we can look at the magnitude of the entry $\bar{B}_{ij}$ which tells us the relative easy by which eigenmode i can be controlled by input j. The entries $\bar{B}_{ij}$ are in fact the transfer functions for the rate of eigenmode i with respect to the input j. These transfer functions for the two bicycle rider models are shown in Fig. 4bc and 5bc for the unstable or partly unstable eigenmodes only.

Some general remarks about these transfer functions. They are clearly zero at the uncontrollable speeds and we see a low gain around these speeds, indicating difficulty to control. It is interesting to see that for the two types of control, steer torque en upper body lean torque, the transfer function have similar shape but different values. The steer torque transfer is larger by a factor of five compared to the upper body lean transfer, indicating easier control by steering than by body lean. The only distinction is the lean mode which is of course easier to control by upper body lean. Furthermore, we see that around double roots, that is were two real eigenvalues coalesce, the transfer functions become very large. This is a defect of the modal method. At such a double root the system can become defective in the sense that the eigenvalues do not span the entire space of the system. Then

Copyright © 2010 by JSME

the inverse transformation $\mathbf{V}$ does not exist and a more general method for defective or near defective systems should be used [10]. Finally, from these transfer functions we see that the for Stratos bicycle configuration, Fig. 4, the unstable weave mode at low speed and the unstable capsize mode at high speed can easily be controlled by either steer torque or upper body lean torque, where steering seems the easier of the two. For the Browser bicycle configuration, Fig. 5, the unstable capsize mode at high speed can easily be stabilized by either steer or upper body lean, where again steering seems easier. However at lower speed it gets more difficult to control this mode irrespective of the type of control. Note that the unstable upper body lean mode, which is present in both models, will be stabilized in the usual manor like in sitting upright at rest, and is of no real concern to the control of the bicycle.

4 Conclusions

We have shown that a Whipple like bicycle with extended rider model can be controlled by either steer torque or upper body lean torque in the practical forward speed range of 1 to 10 m/s. Although the underactuated system has some forward speeds for which the system is uncontrollable, these are either stable modes or at near to zero forward speed, and therefore of no concern. To investigate and compare the controllability around uncontrollable speeds we have introduced the transfer functions of the eigenmode rates with respect to the control inputs. These transfer functions give some insight into the ease of control.

Future work is directed towards the investigation of modal controllability around the double roots where the system is defective or near-defective.

ACKNOWLEDGEMENT

Thanks to Mark Psiaki for pointing out important issues in controllability analysis, to Jason Moore for measuring the bicycles and riders during his Fulbright granted year (2008/2009) at TUDelft, and to Batavus for supplying the bicycles.

References

[1] J. P. Meijaard, Jim M. Papadopoulos, Andy Ruina, and A. L. Schwab. Linearized dynamics equations for the balance and steer of a bicycle: a benchmark and review. *Proceedings of the Royal Society A*, 463:1955–1982, 2007.

[2] F. J. W. Whipple. The stability of the motion of a bicycle. *Quarterly Journal of Pure and Applied Mathematics*, 30:312–348, 1899.

[3] J. K. Moore, J. D. G. Kooijman, and A. L. Schwab. Rider motion identification during normal bicycling by means of principal component analysis. In M. Wojtyra K. Arczewski, J. Fraczek, editor, *MULTIBODY DYNAMICS 2009, ECCOMAS Thematic Conference*, 29 June–2 July 2009, Warsaw, Poland, 2009.

[4] J. D. G. Kooijman and A. L. Schwab. Some observations on human control of a bicycle. In I. Zobory, editor, *11th mini Conference on Vehicle System Dynamics, Identification and Anomalies (VSDIA2008), Budapest, Hungary*, page 8. Budapest University of Technology and Economics, Nov. 2008.

[5] A. L. Schwab and J. D. G. Kooijman. Lateral dynamics of a bicycle with passive rider model. In *The 1st Joint International Conference on Multibody System Dynamics, IMSD2010*, page 9, May 25–27, 2010, Lappeenranta, Finland, 2010.

[6] N. Nagai. Analysis of rider and single-track-vehicle system; its application to computer-controlled bicycles. *Automatica*, 19(6):737–740, 1983.

[7] K. A. Seffen, G. T. Parks, and P. J. Clarkson. Observations on the controllability of motion of two-wheelers. In *Proceedings IMechE*, volume 215 Part I, pages 143–156, 2001.

[8] J. B. Jonker and J. P. Meijaard. SPACAR - Computer program for dynamic analysis of flexible spatial mechanisms and manipulators. In W. Schiehlen, editor, *Multibody systems handbook*, pages 123–143. Berlin, Germany: Springer, 1990.

[9] J. K. Moore, M. Hubbard, J. D. G. Kooijman, and A. L. Schwab. A method for estimating physical properties of a combined bicycle and rider. In *Proceedings of the ASME 2009 International Design Engineering Technical Conferences & Computers and Information in Engineering Conference*, DETC2009, Aug 30 – Sep 2, 2009, San Diego, CA, 2009.

[10] YD Chen, SH Chen, and ZS Liu. Quantitative measures of modal controllability and observability in vibration control of defective and near-defective systems. *Journal of Sound and Vibration*, 248(3):413–426, 2001.

Copyright © 2010 by JSME