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Calspan

*BICYCLE DYNAMICS – SIMPLIFIED STEADY STATE RESPONSE CHARACTERISTICS AND STABILITY INDICES*

**Task Order No. 2**

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This report has been prepared by Roy S. Rice and reviewed and approved by Raymond R. McHenry.

Raymond R. McHenry

Raymond R. McHenry, Asst. Dept. Head
Transportation Safety Department

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TABLE OF SYMBOLS

| | | |
|---|---|---|
| $M$ | - total system (bicycle plus rider) mass, | $\frac{\text{lb-sec}^2}{\text{ft}}$ |
| $M_R$ | - rider upper body mass, | lb-sec$^2$/ft. |
| $T$ | - applied steering torque, | ft-lb. |
| $V$ | - forward velocity | ft/sec |
| $W$ | - total system weight, Mg | lbs. |
| $W_S$ | - steering assembly weight, $M_S g$ | lbs. |
| $Z_F$ | - ground reaction force at front wheel contact patch | lbs. |
| $C_\alpha$, $C_\varphi$ | - tire stiffness coefficients* | lbs/rad. |
| $R$ | - tire rolling radius | ft. |
| $a$ | - horizontal distance between center of front wheel and total system c.g. | ft. |
| $b$ | - horizontal distance between center of rear wheel and total system c.g | ft. |
| $f$ | - front fork mass offset; perpendicular distance from steer axis to steering assembly, c.g | ft. |
| $g$ | - gravitational constant | ft/sec$^2$ |
| $h$ | - vertical distance from road surface to total system c.g. | ft. |
| $i$ | - wheel moment of inertia about its spin axis ($i_F$ - front wheel; $i_T$ - both wheels) | lb-ft-sec$^2$ |
| $k$ | - vertical distance between rider upper body c.g. and the upper body pivot point | ft. |

*For convenience in keeping the signs of $C_\alpha$ and $C_\varphi$ the same, the cornering stiffness coefficient ($C_\alpha$) sign is considered positive, which is opposite to the practice in automobile dynamics.

TABLE OF SYMBOLS (cont.)

| | | |
|---|---|---|
| l | - bicycle wheelbase | ft. |
| r | - yaw rate | rad/sec. |
| t | - mechanical steering trail, perpendicular distance from steer axis to center of front wheel contact patch | ft. |
| $\alpha$ | - tire slip angle | rad. |
| $\varphi$ | - tire inclination (camber) angle | rad. |
| $\beta$ | - bicycle sideslip angle | rad. |
| $\delta$ | - steering assembly displacement angle | rad. |
| $\sigma$ | - steering axis inclination angle | rad. |

See Figure 1.

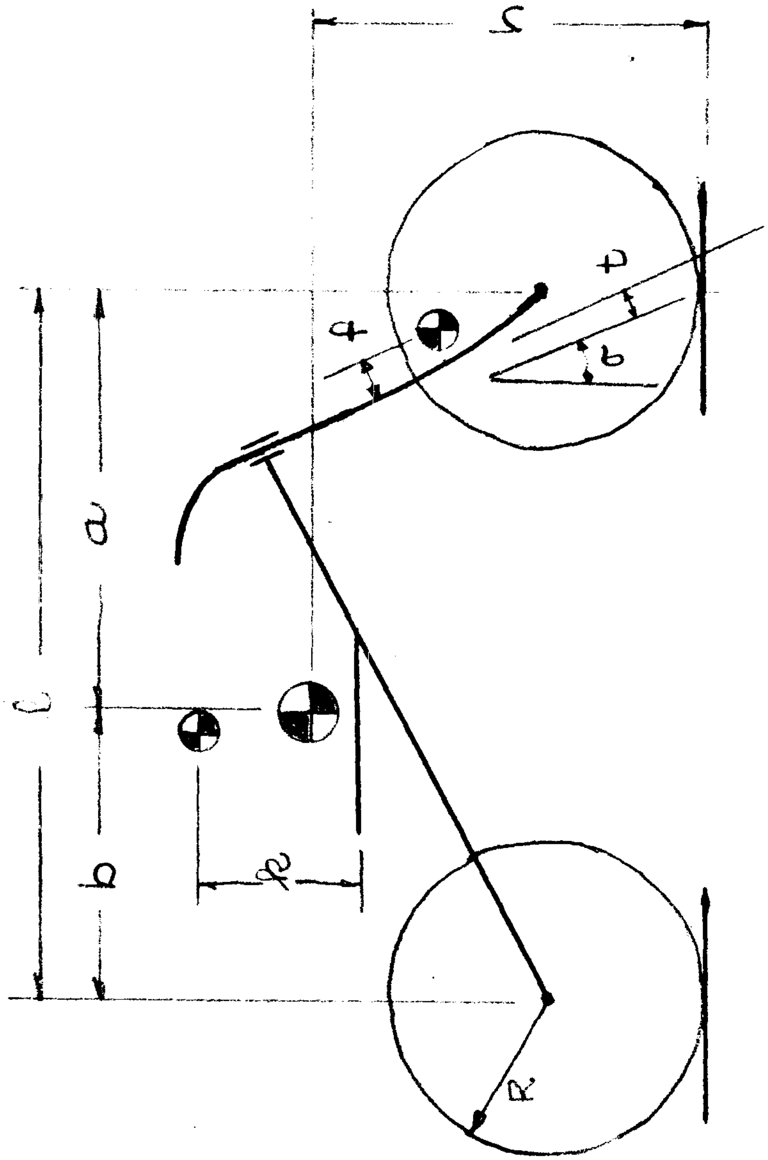


FIGURE 1
CHARACTERISTIC DIMENSIONS OF BICYCLE

## 1.0 INTRODUCTION

From the outset of the Schwinn-sponsored bicycle studies at Calspan, an output of primary interest has been the identification and quantification of performance parameters which can be used to define the stability and controllability characteristics of new bicycle designs. One of the important goals associated with this work is the description of stability indices and the evaluation of these indices, in terms of primary design variables, to show sensitivity to changes in design.

In previous work on the program, the complete nonlinear digital computer simulation has been used for such investigations (References 1 through 3). This approach permits a thorough examination of the dynamic performance characteristics of the bicycle but it is not a convenient method for evaluating its fundamental control characteristics. Therefore, a simplified linear analysis of limited scope (position control*) was undertaken to provide a basis for identifying key stability and control parameters. This work is reported in Reference 3. In this note, the analysis has been extended to develop the steady state response characteristics of the bicycle and rider for all control modes.

It is important that the conditions for which these expressions are applicable be fully understood. They are steady-state equilibrium equations - they describe the consistent sets of conditions for which the forces and moments on the rider-bicycle system are in balance. They do not define sets of conditions for which the system is not in trim (i.e. during transient periods in going from one steady state condition to another). In this sense, they describe the "static" stability of the system and permit evaluation of the response gains (or sensitivities) to control inputs.

*i.e. front wheel position is the control input.

## 2.0 DISCUSSION

### 2.1 Equations of Motion

The control of a two-wheel vehicle may be thought of in terms of three distinct approaches to the application of inputs -

1. Position control. In this mode, the rider generates angular displacements of the front wheel with respect to the frame to provide stabilizing or maneuvering inputs. In steady state, the appropriate force and moment balance relationships must always be satisfied and the rider must therefore always provide the appropriate steer angles to produce this balance. It is assumed that the rider supplies whatever steering torque is required to maintain the desired steering angle.

2. Torque control. In this mode, the rider applies torque to the steering assembly, as required, for stability and control of the vehicle. This is the form of control on which the simulation model is based. The output motions of interest are the same as for position control (yaw rate, roll angle, sideslip angle); in addition, the steer angle response is a key term.

3. Rider lean control. With the bicycle, the operator has not only the steering input for control but can also apply body lean to affect response. The effective input to the system is a roll moment, which is a function of the angular displacement between the rider's upper body and the bicycle's plane of symmetry. Again, the response parameters of interest are yaw rate, roll angle, sideslip angle, and steering angle.

The steady-state equations of motion which may be used to investigate the stability and control characteristics of two-wheel vehicles consist of four force or moment balance expressions - (1) side force; (2) yaw moment; (3) roll moment, and (4) steer torque. They are given in this order in equations 1 through 4 which utilize the symbols given in the Table of Symbols.

SIDE FORCE EQUILIBRIUM EQUATION

$$MVr + C_{\alpha F}\left(\delta \cos \sigma - \beta - \frac{ar}{V}\right) + C_{\varphi F}(\varphi + \delta \sin \sigma) + C_{\alpha R}\left(\frac{br}{V} - \beta\right) + C_{\varphi R}\varphi = 0 \qquad [1]$$

This expression simply says that the side forces developed by the tires (due to either slip angle or inclination angle) must equal the centrifugal force due to path curvature.

YAW MOMENT EQUILIBRIUM EQUATION

$$a\left[C_{\alpha F}\left(\delta \cos \sigma - \beta - \frac{ar}{V}\right) + C_{\varphi F}(\varphi + \delta \sin \sigma)\right] = b\left[C_{\alpha R}\left(\frac{br}{V} - \beta\right) + C_{\varphi R}\varphi\right. \qquad [2]$$

This equation balances the horizontal moments around the c. g. due to front and rear tire forces.

ROLL MOMENT EQUILIBRIUM EQUATION

$$MVrh - MVg\varphi + (Z_F t - W_S f)\delta + \frac{i_T}{R} Vr = 0 \qquad [3]$$

The primary terms are the first two - the centrifugal force effect and the opposing roll angle effect. The next term is associated with steering geometry and the last is a gyroscopic effect. For evaluating the steering input relationships, the rider is assumed to remain in the plane of the bicycle and the expression is therefore equated to zero. When rider lean control is applied, the zero is replaced with a roll moment term which is a function of the angle between the plane of the bicycle and the rider's upper body.

STEER MOMENT EQUILIBRIUM EQUATION

$$T = t\left[C_{\alpha F}\left(\delta \cos\sigma - \beta - \frac{ar}{V}\right) + C_{\varphi F}\left(\varphi + \delta \sin\sigma\right)\right] + M_S f V r + \frac{i_F}{R} V r + \delta\left(Z_F t - W_S f\right)\sin\sigma + \varphi\left(Z_F t - W_S f\right) \qquad [4]$$

The terms on the right side of the equation are, in sequence: the primary front tire/mechanical trail effect; the centrifugal force effect on the steering assembly; the gyroscopic term; the steering geometry effect due to steering displacement, and the steering geometry effect due to roll angle.

For computing the rider lean control parameters, the value of T was taken as zero.

## 2.2 Steady-State Response Parameters

The equations given in the previous section may be solved simultaneously to define the response motions in terms of the control inputs. These are the steady state transfer functions of the system. Only the first three equations are used to obtain the position control parameters; all four are used to define the torque control and rider lean control characteristics. The results of these operations are shown in Tables 1 (steering control) and 2 (rider lean control). The symbols N and $\Delta$ have been used to indicate numerators and denominators of the transfer functions, respectively. Thus, $N^{r}_{\delta, T}$ denotes the numerator of the yaw rate (r) response for both steering position ($\delta$) control and steering torque (T) control; $\Delta_T$ is the denominator for torque control. As shown, the same function of steering position serves as the numerator for torque control (called $N^{\delta}_{T}$) and the denominator for position control (called $\Delta_{\delta}$).

| | |
|---|---|
| $N^{r}_{\delta,T}$ | $V/\ell \left[ \cos\sigma + \frac{C_{\varphi F}}{C_{\alpha F}} \sin\sigma - \frac{Z_F t - W_s f}{Mhg} \left( \frac{C_{\varphi R}}{C_{\alpha R}} - \frac{C_{\varphi F}}{C_{\alpha F}} \right) \right]$ |
| $N^{\varphi}_{\delta,T}$ | $V^2/\ell g \left[ \left( \cos\sigma + \frac{C_{\varphi F}}{C_{\alpha F}} \sin\sigma \right) \left( 1 + \frac{i_T}{MRh} \right) \right] + \frac{Z_F t - W_s f}{Mhg} \left[ 1 - \frac{MV^2 (a C_{\alpha F} - b C_{\alpha R})}{\ell^2 C_{\alpha F} C_{\alpha R}} \right]$ |
| $N^{\beta}_{\delta,T}$ | $1/C_{\alpha R} \Big\{ \cos\sigma \left[ \frac{b}{\ell} C_{\alpha R} + \frac{V^2}{\ell g} \left( 1 + \frac{i_T}{MRh} \right) C_{\varphi R} - \frac{MV^2 a}{\ell^2} \right] + \sin\sigma \left[ \frac{b}{\ell} \frac{C_{\varphi F}}{C_{\alpha F}} C_{\alpha R} + \frac{V^2}{\ell g} \left( 1 + \frac{i_T}{MRh} \right) \frac{C_{\varphi F} C_{\varphi R}}{C_{\alpha F}} - \frac{MV^2 a \, C_{\varphi F}}{\ell^2 C_{\alpha F}} \right] + \frac{Z_F t - W_s f}{Mhg\ell} \left[ a C_{\varphi R} + b \frac{C_{\alpha R}}{C_{\alpha F}} C_{\varphi F} + \frac{MV^2}{\ell} \left( b \frac{C_{\varphi R}}{C_{\alpha F}} - a \frac{C_{\varphi F}}{C_{\alpha F}} \right) \right] \Big\}$ |

TABLE 1

STEERING CONTROL RESPONSE PARAMETERS

| | |
|---|---|
| $N^{\delta}_{T}$ or $\Delta_S$ | $1 - \frac{MV^2(aC_{\alpha F} - bC_{\alpha R})}{\ell^2 C_{\alpha F} C_{\alpha R}} + \frac{V^2}{\ell g}\left(1 + \frac{i_T}{MRh}\right)\left(\frac{C_{\phi R}}{C_{\alpha R}} - \frac{C_{\phi F}}{C_{\alpha F}}\right)$ |
| $\Delta_T$ | $(Z_F t - W_s f)\left(\sin\sigma + \frac{Z_F t - W_s f}{Mhg}\right)\left[1 - \frac{MV^2(aC_{\alpha F} - bC_{\alpha R})}{\ell^2 C_{\alpha F} C_{\alpha R}}\right] + \frac{V^2}{\ell g}(Z_F t - W_s f)\left(1 + \frac{i_T}{MRh}\right)\left(\cos\sigma + \frac{C_{\phi R}}{C_{\alpha R}}\sin\sigma\right) + \frac{V^2}{\ell^2}\left[Mtb + \ell\left(M_s f + \frac{i_F \sin\sigma}{R}\right)\right]\left[\cos\sigma + \frac{C_{\phi F}}{C_{\alpha F}}\sin\sigma\right]$ |

TABLE 1 (cont'd)
STEERING CONTROL RESPONSE PARAMETERS

| | |
|---|---|
| $N^{r}_{\varphi_R}$ | $\frac{V}{\ell}\left(Z_F t - W_s f\right)\left(\cos\sigma + \frac{C_{\varphi F}}{C_{\alpha F}}\sin\sigma\right)$ |
| $N^{\varphi}_{\varphi_R}$ | $-\left\{(Z_F t - W_s f)\sin\sigma\left[1 - \frac{MV^2(aC_{\alpha F} - bC_{\alpha R})}{\ell^2 C_{\alpha F} C_{\alpha R}}\right] + \frac{V^2}{\ell^2}\left[Mtb + \ell\left(M_s f + \frac{i_F \sin\sigma}{R}\right)\right]\left(\cos\sigma + \frac{C_{\varphi F}}{C_{\alpha F}}\sin\sigma\right)\right\}$ |
| $N^{\delta}_{\varphi_R}$ | $(Z_F t - W_s f)\left[1 - \frac{MV^2(aC_{\alpha F} - bC_{\alpha R})}{\ell^2 C_{\alpha F} C_{\alpha R}}\right] + \frac{V^2}{\ell^2}\left[Mtb + \ell\left(M_s f + \frac{i_F \sin\sigma}{R}\right)\right]\left(\frac{C_{\varphi R}}{C_{\alpha R}} - \frac{C_{\varphi F}}{C_{\alpha F}}\right)$ |
| $\Delta_{\varphi_R}$ | $\frac{Mh}{M_R k}\Delta_T$ |

TABLE 2

RIDER LEAN CONTROL RESPONSE PARAMETERS

The rider lean control denominator, $\Delta_{\varphi R}$ , is simply the steering torque denominator, $\Delta_{T}$ , modified by a dimensionless constant.

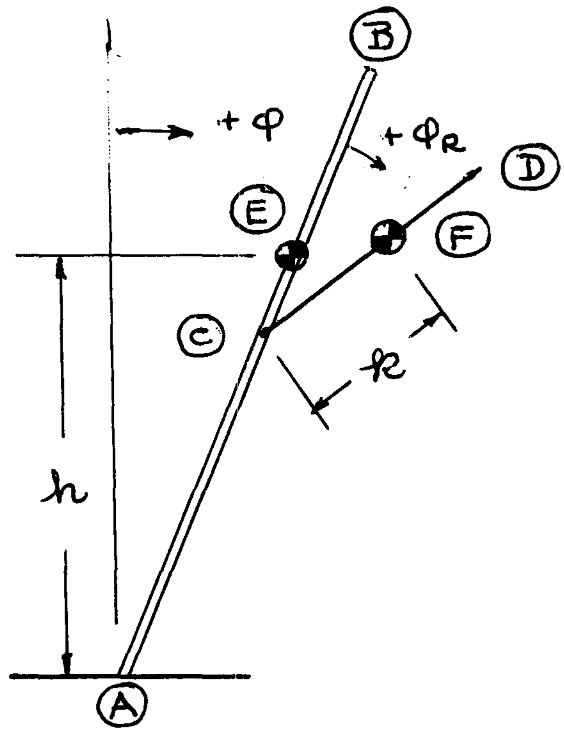
These steady-state solutions describe the set of conditions which must be met for balanced forces and moments. They do not define a condition of dynamic stability nor do they indicate the combination and sequence of inputs necessary to reach the balanced condition. For example, reverse steering torque is necessary for cornering at moderate speeds without rider lean but the initial torque must clearly be in the same direction as the intended displacement.

The steady-state position control transfer functions were previously derived and discussed in Reference 3. They are reproduced, with minor modifications, in Table 1. The primary terms in the numerators of (r) and ( $\varphi$ ) are associated with velocity and wheelbase; although tire characteristics, front end geometry, and gyroscopic effects are present, their roles are minor at reasonable speeds. The principal modifying term is associated with head tube angle - cos $\sigma$ . The sideslip angle ( $\beta$ ) expression is quite involved but most of the terms are small. Note that the rear tire cornering stiffness coefficient is the principal sensitivity term.

The expression for steer angle ( $\delta$ ) is shown as both the numerator of the steering response to a torque input and the denominator for the position control parameters. The terms have been arranged so that it has a basic value of 1 which is then modified by the velocity-sensitive terms.

The expressions for the steady-state torque control parameters are also given in Table 1. The numerators of the transfer functions are the same for torque control as they are for position control; the denominator contains elements which are functions of the steering assembly geometry. The most important factor to be noted is that the sign of the denominator will change at some finite value of speed. The implications of this sign reversal will be discussed in detail in the following section.

The rider lean control transfer functions are given in Table 2. The expressions are based on the additional roll moment applied by the rider in leaning with respect to the plane of the bicycle. This effect can be analyzed with the aid of the following sketch.




The line A-B represents the rigid rider-bicycle system with its c.g. at point E which is at a distance (h) above the ground. For small angles, the clockwise roll torque about A at an inclination angle of $\varphi$ is $M g h \varphi$. Assume that the rider's lower body remains fixed with respect to the bicycle but that his upper body can rotate about a longitudinal axis at point C. The line CD represents the upper body which has its c.g. at point F, a distance (k) above the pivot point. The additive moment due to the rider lean angle, $\varphi_R$ (which is measured with respect to the bicycle) is approximately $M_R g k \varphi_R$. This is clearly a simplification of the true roll moment but it permits the equations of motion to be written in a generalized way that can either include or exclude rider lean control with minimum modification.

The numerators of the r, $\varphi$ and $\delta$ transfer functions for lean control are not the same as those for steering control. The same design parameters are present however, and their effects may be analyzed in the same way. Note that the denominator is the same as that for steering torque control except for the constant multiplying factor.

Prior to discussing the effects of individual design parameters on the performance of conventional bicycles, it is of interest to concentrate on the basic front end geometry effects. A term which recurs throughout these expressions for steady-state performance is ( $Z_F t - W_S f$ ).

The two elements in this term combine front wheel loading ($Z_F$) and mechanical trail (t) considerations and steering assembly weight ($W_S$) and its mass offset (f) from the steering axis, respectively. In conventional bicycle designs, $Z_F > W_S$ and $t > f$; therefore, we can be primarily concerned with the magnitude and sign of the $Z_F t$ portion of this term for these designs. In this case, the product of $Z_F$ (which is always negative) and t (which is positive in the coordinate conventions used in this analysis) is negative. However, the sign of the trail may be negative in certain unconventional designs and, therefore, the sign of $Z_F t$ can be positive.* When this occurs, the values for the speeds at which sign changes occur in the performance parameters become imaginary - that is, there are no sign inversions of the parameters in the normal operational range of speed. Let us examine the meaning of this in terms of stability and control.

One of the primary effects of negative trail (or, more generally, of a positive value of the $Z_F t - W_S f$ term) is that larger roll angles are required for all equivalent cornering conditions. In effect, at a given value of lateral acceleration, the bicycle must be rolled to a larger angle

*It will be noted that when t is small with respect to f (i.e. trail is approximately zero), the effect of $W_S f$ can no longer be neglected.

than a design with positive trail to compensate for the roll moment due to front end geometry which augments the moment due to centrifugal force.

Another effect of negative trail is to decrease the steady state sensitivities of yaw rate and steer angle to input torque - more torque is needed to perform a given cornering maneuver. However, there is no change in sign of this parameter over the speed range.

### 2.2.1 Position Control Parameters

The three position control parameters - $r/\delta$, $\varphi/\delta$ and $\beta/\delta$ - provide information on yaw rate gain, roll sensitivity and tracking, respectively. The first can be extended to $a_y/\delta$, lateral control sensitivity, by multiplying by velocity ( $a_y/\delta = V\ r/\delta$ ). They are useful for describing rider input requirements.

#### Yaw Rate Gain, $r/\delta$

This expression consists of the basic Ackermann steering term plus understeer-oversteer effects due to tire characteristics. The increased responsiveness (i.e., gain) of short wheelbase bicycles is clearly shown by the $V/\ell$ term. It can be said, in general, that high values of this function result in a "nervous" bicycle - it can be more easily overcontrolled. Very low values, on the other hand, result in non-responsiveness; larger steering angles are required to perform a given turn at a given speed.

Note that the value for this parameter depends only very weakly on the camber thrust terms, $C_{\phi F}$ and $C_{\phi R}$. The effect is reduced even further by the small values of these coefficients for high inflation pressure tires in current use. Also, the gyroscopic effect of the wheels, represented by the $i_T$ term, is almost negligible in this parameter. The principal numerator term is therefore the steering head rake effect - $\cos\sigma$. Large values of $\sigma$ (the chopper design) tend to reduce the value of the control gain. In conjunction with the longer wheelbases found in chopper units, the gain can be sufficiently reduced that tight turns at low speed become very difficult to perform.

The primary denominator term, $\left[1 - \frac{MV^2(aC_{\alpha F} - bC_{\alpha R})}{\ell^2 C_{\alpha F} C_{\alpha R}}\right]$ describes the bicycle's understeer-oversteer characteristics. In general, "$aC_{\alpha F}$" and "$bC_{\alpha R}$" are nearly equal (although the first term is usually slightly larger) and bicycles tend to be neutral steer vehicles. In any case, it is undesirable for the speed at which this total term becomes zero to fall in the normal range of operation. In effect,

$$\frac{\ell^2 C_{\alpha F} C_{\alpha R}}{M(aC_{\alpha F} - bC_{\alpha R})} > V_{MAX}^2$$

is a desirable design practice.

Based on the measurements of tire characteristics performed at Calspan (Reference 2), this is almost always the case for matched tire sets. Some combinations, with a tire having a lower $C_\alpha$ value mounted on the rear, can violate this criterion.

Roll Sensitivity, $\phi/\delta$

This parameter provides a convenient measure of the steady-state of the bicycle in a turn. Its denominator is the same as that for the $r/\delta$ transfer function and the same observations as made previously apply. Its numerator is dominated by two principal terms, $\frac{V^2 \cos\sigma}{\ell g}$ and $\frac{Z_F t - W_S f}{Mhg}$, which have opposite signs (for positive mechanical trail). Therefore, a finite speed exists at which the bicycle can be cornered in an upright position (i.e. $\phi = 0$). This speed is

$$V_\phi = \left[\frac{\ell(Z_F t - W_S f)}{Mhg \cos\sigma}\right]^{1/2}$$

For good rideability, it appears to be desirable that this speed be near the lower end of the normal operation speed range. This allows steady-state roll angles to be kept small over the operating range but still provides adequate sensitivity in the roll angle cue at higher speeds where it is important.

It should be noted that no such speed exists for bicycles with negative mechanical trail. For these designs, the sign of the numerator is always positive, producing larger steady state roll angles at all speeds than required by a similar bicycle of conventional design.

It should also be observed that the value for $V_\varphi$ will be higher for a given bicycle if the rider is lightweight, since the values for both (M) and (h) will be reduced.

### Bicycle Sideslip Angle Gain, $\beta/\delta$

This parameter gives a measure of the angle between the bicycle's heading and its velocity vector and therefore indicates how well the unit tracks along curved paths. For reasonable designs, this angle never gets very large under normal riding conditions and it is probably the least significant of the parameters for evaluating rideability. Stripped of terms involving the tire camber thrust coefficients, the expression reduces to $\left(\frac{b}{l} - \frac{MV^2a}{l^2C_{\alpha R}}\right)\cos\sigma$. This shows the basic dependence of the value of this term on bicycle geometry (a, b, l, $C_{\alpha R}$, $\sigma$) which is modified by operational factors - rider weight and speed. Note that the expression in the brackets can be solved for a speed at which the value of $\beta$ will be zero; this is the condition for which the bicycle tracks precisely. Also note that at low speeds the front wheel track is outside that of the rear wheel (the bicycle "noses out") whereas, above the tangent speed (the speed for $\beta = 0$), it "noses in". As indicated earlier, it is not expected that there will be large differences in the value of this parameter among bicycles of reasonable design.

### 2.2.2 Torque Control Parameters

The four torque control parameters - $\psi/T$, $\phi/T$, $\beta/T$, and $\delta/T$ - have the same numerators as their counterpart position control transfer functions have. The $\delta$ denominator becomes the numerator in the $\delta/T$ parameter. The comments made in the previous section with respect to these expressions are still applicable. The $\Delta_T$ function is therefore of primary interest.

The expression consists of terms which all contain steering geometry effects. One of them, ($\sin\sigma - \frac{Z_F t - W_S f}{Mhg}$), is independent of speed; all others are functions of $V^2$. The constant term has opposite sign (in conventional bicycle designs) to the other terms and, therefore, a specific velocity at which the values for all the transfer functions become infinite can be computed. This is a very important stability-related parameter to which we have given the name, Inversion Speed. The sign of the expression changes at this speed; in conventional bicycle designs, it is negative below this speed and positive above it.

It is convenient to revise the expression for $\Delta_T$ in order to evaluate this inversion speed. Note that -

$$Z_F = Mg\,b/\ell \quad ;$$

it is the portion of the total weight which is reacted at the front wheel contact patch. Similarly, $W_F = M_F g$. Therefore,

$$-\frac{Z_F t - W_S f}{\ell g} \doteq \frac{M t b + \ell M_S f}{\ell^2}$$

and several of the terms shown in the expression for $\Delta_T$ are cancelling. The velocity-sensitive portion of the expression may then be approximated (neglecting the terms of the form of $\frac{C_\phi}{C_\alpha}$) and rewritten as:

$$\frac{V^2}{\ell}\left[\frac{(Z_F t - W_S f)}{g}\,\frac{i_T}{MRh} + \frac{i_F \sin\sigma}{R}\right]\cos\sigma$$

For $i_T = 2\, i_F$ and with further clearing, we obtain -

$$\frac{V^2 i_F}{R\ell} \cos\sigma \left[ -2\left(\frac{tb}{h\ell} + \frac{W_s f}{Wh}\right) + \sin\sigma \right]$$

This may now be combined with an abbreviated form of the first term of the expression to give -

$$\Delta_T = (Z_F t - W_s f)\left(\sin\sigma - \frac{tb}{h\ell} \quad \frac{W_s f}{Wh}\right) + \frac{V^2 i_F}{\ell R} \cos\sigma \left[ \sin\sigma - 2\left(\frac{tb}{ht} - \frac{W_s f}{Wh}\right) \right]$$

If $\left(\frac{tb}{h\ell} - \frac{W_s f}{Wh}\right)$ is small with respect to sin $\sigma$ (a reasonable condition for conventional design), then the inversion speed ($V_I$) can be approximated by -

$$V_I^2 = \frac{(Z_F t - W_s f)\, \ell R}{i_F \cos\sigma}$$

This approximation gives estimates of $V_I$ which are slightly low but it does show the importance of the value of trail to this index for a given size bicycle and the significance of the gyroscopic effect of the front wheel.

### 2.2.3 Rider Lean Control Parameters

Although rider lean control is normally associated with transient maneuvering, it may also be part of the steady-state control technique used by the rider. In fact, it is the only control available to the rider operating hands-off. The steady-state transfer functions given in Table 2 offer a means for evaluating the performance of the system in this mode of control and they give some insight into the riding operation when this mode is used to augment or offset the effects of applied steering torque.

Three rider lean control parameters - $r/\varphi_R$, $\varphi/\varphi_R$, and $\delta/\varphi_R$ have been identified for further analysis. The denominators of these terms are the same as for steering torque control (except for the constant multiplier) but the numerators are different. Hence, the gains (sensitivities) are changed and, in the case of $\varphi/\varphi_R$, a new speed for which the numerator value is zero is defined. Also note that the basic sign of this parameter is negative.

As indicated previously, the rider lean control equations of motion were written for the condition in which applied steering torque is zero. In effect, this is the hands-off condition and the steering assembly seeks a displacement for which this torque balance is satisfied. This may not be a stable riding condition in some circumstances.

### 2.2.4 Further Simplifications

The expressions given in Tables 1 and 2 can be further simplified if their application is restricted to more-or-less conventional designs and reasonable speed ranges. In effect, the camber thrust terms ($C_\varphi$) and some of the gyroscopic terms can be discarded. The resultant expressions, given in the form of the complete parameters rather than in the numerator and denominator form of the previous tables, are shown in Table 3.

## 2.3 The Specific Speed Terms

The special conditions of operation for which any of the response parameters takes a zero or infinite value will be briefly discussed in this section. These conditions, which are associated with specific values of forward speed, constitute a set of quantifiable indices which can be used in evaluating bicycle designs. Five of these speeds will be identified - the critical (or characteristic) speed for yaw stability, the zero sideslip speed for tracking fidelity, two upright cornering speeds for zero roll angle, and the inversion speed at which control torque requirements change sign.

| INDEX | |
|---|---|
| Position Control Yaw Rate Gain. | $\dfrac{V/l \cos\sigma}{1 - \dfrac{MV^2(aC_{\alpha F} - bC_{\alpha R})}{l^2 C_{\alpha F} C_{\alpha R}}}$ |
| Position Control Roll Angle Sensitivity. | $\dfrac{\dfrac{V^2}{lg}\cos\sigma - \dfrac{bt}{lh}\left(1 + \dfrac{W_s l f}{W b t}\right)}{1 - \dfrac{MV^2(aC_{\alpha F} - bC_{\alpha R})}{l^2 C_{\alpha F} C_{\alpha R}}}$ |
| Position Control Sideslip Angle Sensitivity. | $\dfrac{\dfrac{1}{C_{\alpha R}}\left[\dfrac{b}{l} C_{\alpha R} - \dfrac{MV^2 a}{l^2}\right]\cos\sigma}{1 - \dfrac{MV^2(aC_{\alpha F} - bC_{\alpha R})}{l^2 C_{\alpha F} C_{\alpha R}}}$ |
| Torque Control Yaw Rate Gain. | $\dfrac{\dfrac{V}{l}\cos\sigma}{\left[(Z_F t - W_s f) + \dfrac{V^2 i_F}{lR}\cos\sigma\right]\sin\sigma}$ |
| Torque Control Roll Angle Sensitivity. | $\dfrac{\dfrac{V^2}{lg}\cos\sigma - \dfrac{bt}{lh}\left(1 + \dfrac{W_s l f}{W b t}\right)}{\left[(Z_F t - W_s f) + \dfrac{V^2 i_F}{lR}\cos\sigma\right]\sin\sigma}$ |

TABLE 3

REDUCED STEADY-STATE RESPONSE PARAMETERS

| | |
|---|---|
| Torque Control Steering Gain. | $\dfrac{1 - \dfrac{MV^2(aC_{\alpha F} - bC_{\alpha R})}{\ell^2 C_{\alpha F} C_{\alpha R}}}{\left[(Z_F t - W_s f) + \dfrac{V^2 i_F}{\ell R}\cos\sigma\right]\sin\sigma}$ |
| Lean Control Yaw Rate Gain. | $\dfrac{(Z_F t - W_s f)\ M_R k}{Mh} \times \left(\dfrac{r}{T}\right)$ |
| Lean Control Roll Angle Sensitivity. | $\dfrac{(Z_F t - W_s f)\left[1 - \dfrac{V^2}{\ell g \tan\sigma}\left(1 + \dfrac{\ell i_F \sin\sigma}{MRbt + M_s R f \ell}\right)\right]}{\dfrac{Mh}{M_R k}\left[(Z_F t - W_s f) + \dfrac{V^2 i_F}{\ell R}\cos\sigma\right]}$ |
| Lean Control Steering Gain. | $\dfrac{(Z_F t - W_s f)\ M_R k}{Mh} \times \left(\dfrac{\delta}{T}\right)$ |

TABLE 3 (cont.)

Critical/Characteristic Speed, $V_C$

This term, which is carried over from automobile practice, is a metric describing the yaw stability of the bicycle. It is defined by the expression -

$$V_C^2 = \frac{\ell^2 C_{\alpha F} C_{\alpha R}}{M(aC_{\alpha F} - bC_{\alpha R})}$$

and it is primarily a function of weight distribution and tire cornering stiffness. The sign of the factor ($aC_{\alpha F}$ - $bC_{\alpha R}$ ) designates whether the computed speed is the critical speed - the speed at which the bicycle becomes statically unstable - or the characteristic speed - the speed at which the $\delta$ value is twice that at zero speed for a given turn. The negative sign is associated with characteristic speed; a positive value indicates that a critical speed exists. In general, these speeds are well outside of the normal operating speed range of the bicycle although the effect may influence sensitivities within the operating range to some degree.

Zero Roll Angle Speed for Steering Control, $V_\varphi$

In the absence of other significant effects, this speed is primarily determined from -

$$V_\varphi^2 = \frac{(Z_F t - W_S f)\ell}{M h \cos\sigma}$$

For reasonable designs, it will have a value of a few feet per second. Note that for low values of trail and mass offset, the value for this speed index tends toward zero.

Zero Roll Angle Speed for Rider Lean Control, $V_{\varphi R}$

This speed can be presented approximately by -

$$V_{\varphi R}^2 = \frac{\ell g \tan\sigma}{\left[1 - \frac{i_F \ell \sin\sigma}{R(Mtb + M_S f \ell)}\right]}$$

for bicycles with positive trail. For reasonable designs, it will have a value of several feet per second. At speeds below $V_{\phi R}$, the signs of the bicycle roll angle and rider lean angle are opposite; above $V_{\phi R}$, these variables have the same sign (up to the inversion speed).

Zero Slip Angle Speed for Steering Control, $V_\beta$

To first order, this speed is defined by -

$$V_\beta^2 \simeq \frac{b \, l \, C_{\alpha R}}{M a}$$

For normal bicycle designs, the value of this index is about 15-20 ft/sec. (10-15 mph). Throughout most of the operating speed range, bicycles with values of this index on this order will track with little sideslip. Designs with short wheel bases and/or lower cornering stiffness rear tires will have lower values for $V_\beta$.

Inversion Speed, $V_I$

The most significant of the stability index speeds is the one which is associated with a zero value of the denominator term for steering torque and rider lean control. It identifies the operating condition for which, in fact, no steady state in r, $\phi$, and $\delta$ is achievable. As indicated by the expressions given in Tables 1 and 3, this speed is -

$$V_I^2 = \frac{(Z_F t - W_S f)\left[\sin\sigma + \dfrac{Z_F t - W_S f}{M h g}\right]}{\dfrac{i_F \cos\sigma}{l R}\left[\sin\sigma + \dfrac{2(Z_F t - W_S f)}{M h g}\right]}$$

The expression involves terms which are functions of the steering system geometry.

Physically, this index identifies operating conditions for which a heavy burden is placed on the rider for maintaining system stability and controllability. For conventional bicycles, the inversion speed is of the order of 15 to 20 mph, but designs with short trail will have smaller values which will make them difficult to ride.

## 2.4 Steady-State Stability Indices

The performance parameters discussed in the previous section provide some insight into the definition of quantitative measures of bicycle responses which may be employed as Stability Indices. In addition, it has been shown that some of these parameters may have zero or infinite values at specific values of forward speed and that these speeds are also significant in characterizing stability and/or controllability. In fact, it appears that if real and finite speeds of this kind do not exist in the design, the bicycle may be uncontrollable in the sense of steady-state response in certain operational ranges.

Three bicycle designs have been compared on the basis of their values for several of the response parameters developed in this analysis. The results are reported in Reference 5 , which has been previously submitted to Schwinn. Table 1 of that reference is reproduced here as Table 4 to illustrate some typical results. Briefly, the table which contains the values for the parameters at only one speed (10 mph), shows that relatively little difference in performance is to be expected between a proven Schwinn design and a so-called maximum trail design but that a "minimum trail design" would be distinctly different in the normal speed range. Although we are not yet able to evaluate the significance of small differences in the values of the parameters with respect to design optimization (for example, we cannot say that the differences between the Schwinn

| | Symbol | Dimension | STD. Suburban | Max. Trail | Min. Trail |
|---|---|---|---|---|---|
| Position Control Yaw Rate Gain | $r/\delta$ | $\frac{deg/sec}{deg}$ | 4.05 | 3.94 | 4.18 |
| Position Control Roll Sensitivity | $\phi/\delta$ | deg/deg | 1.84 | 1.79 | 1.92 |
| Position Control Sideslip Sensitivity | $\beta/\delta$ | deg/deg | .23 | .23 | .24 |
| Torque Control Yaw Rate Gain | $r/T$ | $\frac{deg/sec}{in-lb}$ | -.069 | -.042 | 1.47 |
| Torque Control Roll Sensitivity | $\phi/T$ | deg/in-lb | -.051 | -.019 | .68 |
| Torque Control Steering GLW | $\delta/T$ | deg/in-lb | -.017 | -.011 | .35 |
| Lean Control Yaw Rate Gain | $r/\phi_R$ | $\frac{deg/sec}{deg}$ | 3.89 | 2.94 | -25.2 |
| Lean Control Roll Sensitivity | $\phi/\phi_R$ | deg/deg | 1.55 | 1.13 | -11.8 |
| Lean Control Steering Gain | $\delta/\phi_R$ | deg/deg | .95 | .74 | -5.98 |
| Inversion Speed | $V_I$ | mph | 17.5 | 19.5 | 9.6 |
| Zero Roll Speed (Steer) | $V_\phi$ | mph | 1.2 | 1.4 | .4 |
| Zero Roll Speed (Lean) | $V_{\phi_R}$ | mph | 4.2 | 4.7 | 3.3 |

TABLE 4

STABILITY INDEX COMPARISON

Suburban and the "maximum trail design" are meaningful to the rider), we are confident that these steady state indices are in fact useful for stability and controllability evaluations.

The values given in the table are for the condition of a 160 pound rider in the upright riding position at 10 mph. This speed is very close to the inversion speed of the minimum trail design for this condition and this configuration therefore exhibits marked differences in the values of the performance metrics from the other two designs for both steering torque control and rider lean control. The very high values for the sensitivity parameters at this condition suggest that this bicycle would be quite difficult to ride in the normal speed range since it would require excessive rider attention.

## 3.0 CONCLUSIONS

The steady-state bicycle Response Parameters developed in this analysis, together with the definition of critical operating conditions which we have called Stability Indices, provide a partial basis for evaluating bicycle stability and control in quantitative terms. It has been shown that the values of some of these terms provide clear differentiation among bicycles of various designs in a performance sense.

In evaluating the significance of these performance parameters, it is necessary that general subjective criteria of desirable characteristics be established. They are:

1. The bicycle should corner with reasonable roll angle throughout its nominal operating speed range.

2. The bicycle should corner with minimum sideslip angle (i.e. the angular difference between the velocity vector and heading should be small).

3. The bicycle should not be overly sensitive to any mode of control input throughout its nominal operating speed range.

On the basis of these criteria, the significant stability indices and response parameters are:

- Position control sideslip angle sensitivity
- Zero roll angle speed for rider lean control
- Inversion speed

The values for these indices depend primarily of front end geometry (rake angle and trail), front wheel gyroscopic effect, and rear tire cornering stiffness for a given bicycle size and rider weight. These design factors appear to be the critical elements for assuring satisfactory stability and control.

## 4.0 RECOMMENDATIONS

The response parameters and their variants, the stability indices, identified above provide a basis for evaluating and comparing bicycles. In some cases, the magnitude and sign of the term may be directly significant; in others, however, there is no supporting data to put the value of the term in perspective. For example, it is difficult to predict whether a difference, of say, 10% in the value of $r/\delta$ at some given speed for two different bicycles is meaningful to the rider. We therefore recommend the following:

1. Perform full-scale tests on a few selected instrumented bicycles to validate the computations and obtain comparative values for some of the response parameters. In effect, we recommend initiating work on Task 4 of the program which was submitted to Schwinn in September 1973 (Reference 6 ). The tests would include riding in constant radius circles at several speeds under both steering control and hands-off. Measured data should include speed, steer angle, steer torque, roll angle, rider lean angle. In addition, we would want to examine correlations of subjective opinions of the riders regarding stability and control with values of the indices.

2. It would be very convenient in performing these tests to have a variable characteristics bicycle so that substantial changes could be made in the values of the design parameters and thereby examine performance extremes. We therefore recommend that Task 3 of the proposed program also be initiated.

3. The numerical results given here have been concentrated on comparing steering geometry designs. It is also of interest

to examine the effects of other design factors, rider weight, and riding position on these steady state control indices. We recommend that a brief parametric study of these factors be initiated.

4. The analyses given here can be extended to yield simplified representations for the dynamic stability characteristics of bicycles. This was done to a limited degree for the position control mode of operation in Reference 3. We now recommend that a similar analysis be made of the characteristic expression for the steering torque control and rider lean control modes. The output of this work would provide additional performance indices (characteristic time constants, frequencies, and damping factors) to supplement the steady state sensitivity parameters developed here.

## 5.0 REFERENCES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