CONTROLLED AND PROGRAMMED MOTION OF A BICYCLE ON A PLANE

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Vol. 13, No. 6, pp. 22-28, 1978

UDC 531.384

It is well known that the bicycle is the basic kinematic element of single-track vehicles (airplane with a bicycle undercarriage, motorcycle, and so on). It consists of a frame and two coupled wheels, one of which can be steered. Classical results on the kinematics and dynamics of the bicycle were developed and generalized in [1]. Among more recent papers, we note [2-5]. All these papers are concerned with the uniform motion of the linear model.

In this paper, we assume the realization of the classical nonholonomic constraints imposed on the rolling of a disc, and discuss a bicycle of general design in the case of small inclination to the vertical and small steering angles. We consider the accelerated motion of the bicycle on a trajectory of variable but sufficiently small curvature. We examine special cases of programmed and controlled motion, including the programmed rolling of a bicycle into a circular trajectory and investigate the reaction of the bicycle to steering for different combinations of the parameters of the motion. We estimate the slowing down of the bicycle on a curvilinear trajectory of increasing curvature and discuss variables that are practically independent of this effect.

1. Following [1], we describe the configuration of the bicycle (Fig. 1a) in terms of the coordinates $x, y, \theta, \chi, \psi, \vartheta_1, \vartheta_2$. The equations for the nonholonomic constraints are ($R=K_1M_1=K_2M_2=K_2'M_2'$ is the wheel radius)

$$x^{\cdot}\cos\theta+y^{\cdot}\sin\theta-R\vartheta_1^{\cdot}=0,\qquad -x^{\cdot}\sin\theta+y^{\cdot}\cos\theta=0 \tag{1.1}$$

$$x'^{\cdot}\cos\theta'+y'^{\cdot}\sin\theta'-R\vartheta_2^{\cdot}=0,\qquad -x'^{\cdot}\sin\theta'+y'^{\cdot}\cos\theta'=0 \tag{1.2}$$

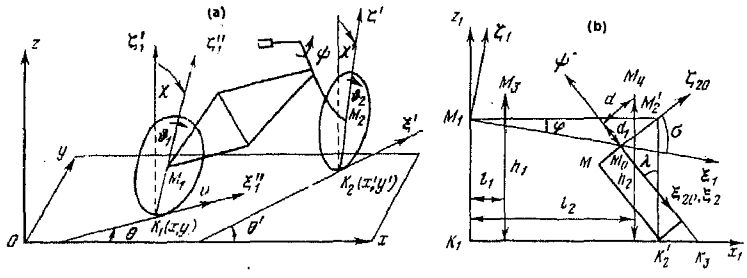

Fig. 1

If we introduce the quasi-velocities $v=x^{\cdot}\cos\theta+y^{\cdot}\sin\theta$, $U=-x^{\cdot}\sin\theta+y^{\cdot}\cos\theta$, we can rewrite (1.1) in the form $v=R\vartheta_1^{\cdot}$, $U=0$.

Let x_1, y_1, z_1 be the coordinates of the center M_1 of the back wheel in the stationary frame $Oxyz$. We then have $x_1=x+R\sin\theta\sin\chi$, $y_1=y-R\cos\theta\sin\chi$, $z_1=R\cos\chi$. If we rotate the x and y axes through the angle θ about the z axis, we obtain the coordinate frame $\xi_1'\eta_1'\zeta_1'$. Rotation of the latter through the angle χ about the ξ_1' axis gives the orientation of the system $K_1\xi_1''\eta_1''\zeta_1''$. Let M_0 be the point of intersection between the steering axis and the perpendicular dropped onto this axis from the center M_2 of the front wheel (Fig. 1b). Let φ be the angle between the line M_1M_0 and the horizontal. The direction of the line drawn from

M_1 to M_0 will be taken to be the positive direction of the axis $M_1\xi_1$; $M_1\eta_1'' \| M_1\eta_1$; $M_1\xi_1\zeta_1$ is the plane of the frame. We now introduce the further frame $M_0\xi_{20}\eta_{20}\zeta_{20}$ so that $M_0\eta_{20} \| M_1\eta_1$, $(\xi_{20} \hat{,} \xi_1) = \pi/2 - \sigma$, $(\xi_1 \hat{,} \zeta_{20}) = \sigma$. Rotation of the axes $M_0\eta_{20}$, $M_0\zeta_{20}$ about the $M_0\xi_{20}$ axis through the angle ψ gives the frame $M_0\xi_2\eta_2\zeta_2$. In Fig. 1b, M_2', K_2' represents the positions of the points M_2, K_2 for $\psi = 0$. We now take

$$\lambda = \sigma - \varphi, \quad M_1M_0 = a, \quad M_0M_2 = b, \quad MM_0 = c_1, \quad K_2'K_3 = \tau, \quad K_1K_2' = c$$
$$R_0 = R - a \sin\varphi + b \cos\psi \sin\lambda, \qquad c_0 = a \cos\varphi + b \cos\psi \cos\lambda$$

so that

$$x_2 = x + (R_0 \sin\chi - b \sin\psi \cos\chi) \sin\theta + c_0 \cos\theta$$
$$y_2 = y - (R_0 \sin\chi - b \sin\psi \cos\chi) \cos\theta + c_0 \sin\theta, \qquad z_2 = R_0 \cos\chi + b \sin\psi \sin\chi$$

where x_2, y_2, z_2 are the coordinates of M_2 in $Oxyz$. The following geometric relationships are valid [1]:

$$a \sin\varphi = b \sin\lambda, \qquad c = a \cos\varphi + b \cos\lambda, \qquad c_1 = R \sin\lambda - b$$

Next, we have

$$x' = x - R \sin\theta' \sin\chi' + (R_0 \sin\chi - b \sin\psi \cos\chi) \sin\theta + c_0 \cos\theta$$
$$y' = y + R \cos\theta' \sin\chi' - (R_0 \sin\chi - b \sin\psi \cos\chi) \sin\theta + c_0 \sin\theta$$

$$R \cos\chi' = R_0 \cos\chi + b \sin\psi \sin\chi$$
$$\sin\theta' \cos\chi' = (\cos\psi \cos\chi + \sin\psi \sin\chi \sin\lambda) \sin\theta + \sin\psi \cos\lambda \cos\theta$$
$$\cos\theta' \cos\chi' = (\cos\psi \cos\chi + \sin\psi \sin\chi \sin\lambda) \cos\theta - \sin\psi \cos\lambda \sin\theta$$
$$\sin\chi' = \sin\chi \cos\psi - \sin\psi \cos\chi \sin\lambda, \qquad \chi' = \chi - \psi \sin\lambda + \ldots$$
$$\dot{\theta}' = \dot{\theta} + \dot{\psi} \cos\lambda + \ldots$$

Equations (1.2) now assume the form

$$\dot{x} \cos\theta + \dot{y} \sin\theta - R\dot{\vartheta}_2 + (c_1 + c \cos\lambda)\dot{\theta}\psi + \ldots = 0$$
$$(\dot{x} \cos\theta + \dot{y} \sin\theta)\, \psi \cos\lambda + c_1\dot{\psi} - c\dot{\theta} + \ldots = 0 \tag{1.3}$$

For the quasi-velocities $\omega_1, \ldots, \omega_4$ we take the left-hand sides of (1.1) and (1.3); $\omega_5 = v$, $\omega_6 = \dot{\psi}$, $\omega_7 = \dot{\chi}$. The inverted relationships are

$$\dot{x} = -\omega_2 \sin\theta + \omega_5 \cos\theta, \quad \dot{y} = \omega_2 \cos\theta + \omega_5 \sin\theta, \quad \dot{\theta} = (-\omega_4 + \omega_5\psi \cos\lambda + c_1\omega_6)c^{-1} + \ldots$$
$$\dot{\vartheta}_1 = (-\omega_1 + \omega_5)R^{-1}, \quad \dot{\vartheta}_2 = R^{-1}\{-\omega_3 - \omega_4\psi(\cos\lambda + c_1/c) + \omega_5[1 + (1/2\cos^2\lambda + c_1c^{-1}\cos\lambda -$$
$$- c_1/2R^{-1}\sin\lambda)\psi^2 + c_1R^{-1}\psi\chi] + \omega_6[(c_1^2c^{-1} + c_1\cos\lambda + R\sin\lambda\cos\lambda)\psi -$$
$$- \chi R\cos\lambda] - \omega_7\psi R\cos\lambda\} + \ldots$$

Here and above the repeated dots represent terms that do not participate in the formation of the linearized (in ψ, χ and their time derivatives) equations of motion. We shall now use the Boltzmann-Hamel method to set up by the dynamic equations [6,7]. If we reject the nonlinear terms, we obtain

$$\frac{d}{dt}\frac{\partial T}{\partial v} = P_5, \quad \frac{d}{dt}\frac{\partial T}{\partial \dot{\psi}} - \frac{\partial T}{\partial \psi} +$$
$$+ v\left[\frac{c_1}{c}\frac{\partial T}{\partial \omega_2} + c_1\left(\frac{\sin\lambda}{R} - \frac{\cos\lambda}{c}\right)\psi\frac{\partial T}{\partial \omega_3} - \frac{\partial T}{\partial \omega_4}\cos\lambda\right] = P_6$$
$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\chi}} - \frac{\partial T}{\partial \chi} - \frac{c_1}{R}v\psi\frac{\partial T}{\partial \omega_3} = P_7$$

It can be shown that

$$M\dot{v} = P_5, \quad a_0\ddot{\chi} - a_1\chi - a_2\ddot{\psi} - a_3v\dot{\psi} + (a_4 - a_5v^2 + a_6\dot{v})\psi = 0$$
$$b_0\ddot{\psi} + (b_1v + \delta)\dot{\psi} + (b_2v^2 - b_3 + b_4\dot{v})\psi - a_2\ddot{\chi} + b_5v\dot{\chi} + (a_4 + b_6\dot{v} + b_7v^2)\chi = 0 \tag{1.4}$$

where $M = m_1 + m_2 + m_3 + m_4 + (I_1 + I_2)R^{-2}$ and P_5 is the linearized generalized force referred to the quasi-coordinate π_5 and including the torque applied to the front wheel by the motor, the frictional forces on the wheel axes, air resistance, and rolling friction, and

$$a_0 = J_1, \quad a_1 = g\nu_1, \quad a_2 = J_3 + c_1/cJ_{12}, \quad a_3 = \nu\cos\lambda + c_1\mu_2, \quad a_4 = g\mu_1$$
$$a_5 = \mu_2\cos\lambda, \quad a_6 = -\nu\cos\lambda, \quad b_0 = J + c_1^2c^{-2}J_2 + 2c_1c^{-1}J_4$$
$$b_1 = (c_1c^{-1}J_2 + J_4)c^{-1}\cos\lambda + c_1^2c^{-2}\nu_3 + c_1c^{-1}\mu_3 + m_2(b + c_1)\cos\lambda, \quad b_2 = (\mu_4 + c_1c^{-1}\nu_3)c^{-1}\cos\lambda$$
$$b_3 = g\mu_1\sin\lambda, \quad b_4 = (c_1c^{-1}J_2 + J_4)c^{-1}\cos\lambda +$$
$$+ c_1[(m_2 + I_2R^{-2})\cos\lambda + c_1c^{-1}I_2R^{-2}] - c_1c^{-1}\mu_5 - m_4d\cos\lambda$$
$$b_5 = \nu_2, \quad b_6 = c_1\mu_2, \quad b_7 = I_2R^{-2}c_1, \quad \nu = m_2R + I_2R^{-1} + J_{12}c^{-1}$$
$$\nu_1 = (m_1 + m_2)R + m_3h_1 + m_4h_2, \quad \nu_2 = I_2R^{-1}(c_1c^{-1} + \cos\lambda) + c_1c^{-1}I_1R^{-1}$$

$$v_3=m_3l_1+m_4l_2,\quad \mu=m_2+m_4+I_2R^{-2},\quad \mu_1=m_2b+m_4d$$
$$\mu_2=[(I_1+I_2)R^{-1}+v_1]c^{-1},\quad \mu_3=\mu_1+m_2c_1,\quad \mu_4=\mu_3+I_2R^{-1}\sin\lambda,\quad \mu_5=\mu_1+I_2R^{-1}\sin\lambda$$
$$J=A_2\sin^2\lambda+B_2\cos^2\lambda+D_2\sin 2\lambda+I_2'+m_4d^2+m_2(b+c_1)^2$$
$$J_1=A_1+A_2+I_1'+I_2'+(m_1+m_2)R^2+m_3h_1^2+m_4h_2^2,\quad J_2=B_1+B_2+I_1'+I_2'+m_3l_1^2+m_4l_2^2$$
$$J_3=(A_2+I_2')\sin\lambda+D_2\cos\lambda+m_2R(b+c_1)+m_4h_2d,\quad J_4=(B_2+I_2')\cos\lambda+D_2\sin\lambda+m_4l_2d$$
$$J_{12}=D_1+D_2+m_3h_1l_1+m_4h_2l_2,\quad h_2=R-a\sin\varphi+d_1\cos\lambda+d\sin\lambda,\quad l_2=a\cos\varphi-d_1\sin\lambda+d\cos\lambda$$

In these expressions m_1, I_1, I_1' is the mass and axial and diametral moments of inertia of the back wheel, m_2, I_2, I_2' are the same quantities for the front wheel, m_3, M_3 are the mass and the center of mass of the frame, A_1, B_1, D_1 are the moments of inertia and the centrifugal moment of inertia of the frame about the axes $M_3\xi_1''$, $M_3\zeta_1''$, m_4, M_4 is the mass and the center of mass of the steering mechanism, A_2, B_2, D_2 are the moments of inertia and the centrifugal moment of inertia of the steering handles relative to the M_4x_2, M_4z_2 axes, $(\xi_2\hat{,}\,x_2)=\pi/2-\lambda$, $(x_2\hat{,}\,\zeta_2)=\lambda$, and δ is the coefficient of viscous friction in the steering system.

Let k_1, R_1, s_1 be the curvature, radius of curvature, and arc length of the trajectory of the point K_1 and k_2, R_2, s_2 the analogous quantities for the point K_2 on the trajectory, i.e., $k_1=d\theta/ds_1=R_1^{-1}$, $k_2=d\theta'/ds_2=R_2^{-1}$. We then have $\theta^{\cdot}=k_1v$, $\theta'^{\cdot}=k_2v'$. We also have

$$v=R\vartheta_1^{\cdot},\quad v'=R\vartheta_2^{\cdot},\quad dx=ds_1\cos\theta,\quad dy=ds_1\sin\theta,\quad dx'=ds_2\cos\theta'$$
$$dy'=ds_2\sin\theta',\quad ds_1=vdt,\quad ds_2=v'dt$$

In the linear approximation,

$$\theta^{\cdot}=c_1c^{-1}\psi^{\cdot}+c^{-1}v\psi\cos\lambda,\quad dx=dx'=ds_1=ds_2,\quad v=v',\quad k_1=c^{-1}\psi\cos\lambda+c_1c^{-1}v^{-1}\psi^{\cdot}$$
$$k_2=k_1+v^{-1}\psi^{\cdot}\cos\lambda,\quad x'=x+c,\quad y'=y+c\theta-c_1\psi,\quad \theta'=\theta+\psi\cos\lambda,\quad dy=\theta ds_1,\quad dy'=\theta' ds_2$$

Remark. Let l_0, h_0 be the coordinates of the center of mass P of the bicycle in the frame $K_1x_1z_1$. If $\lambda=0$, $b=0$, $D_2=0$, $d=0$ and the entire mass of the bicycle is localized at P (this is the simplest model), then the first equation in (1.4) yields

$$ch_0\chi^{\cdot\cdot}=cg\underline{\chi}+vl_0\psi^{\cdot}+(v^2+v^{\cdot}l_0)\psi$$

which corresponds to the elementary theory and generalizes the well-known equation [1,8] to the case where $v=v(t)$.

2. Consider the motion of the bicycle without acceleration (v = const) with constant tilt $(\chi=\text{const})$ and fixed steering angle $(\psi=\text{const})$. In this case

$$\theta=\theta_0+vc^{-1}\psi t\cos\lambda,\quad \chi=(a_4-a_5v^2)a_1^{-1}\psi$$

The point K_1 moves on the circle $x^2+y^2=c^2\psi^{-2}\cos^{-2}\lambda$. The values of the radius $R_1=c\psi^{-1}\cos^{-1}\lambda$(m) of the circle for different angles ψ(rad) and c = 1.4 m, λ = 0.4363 rad are given below:

ψ	0.01	0.02	0.03	0.04	0.05	0.06	0.08	0.11	0.15	0.20	0.23	0.26
R_1	155	77.2	51.5	38.6	30.9	25.8	19.3	14.0	10.3	7.72	6.72	5.94

Figure 2 shows the family of straight lines $\chi_1=\chi_1(\psi)$, $\chi_1=-\chi$ with v as a parameter. The values of this parameter, ranging from 2 to 100 m/sec, are indicated against the straight lines. The scale on the horizontal axis is different in the four quadrants: points 0,2,4,... correspond to the first quadrant, the points 0,0.01,0.02 correspond to the fourth quadrant, and the second and third quadrants are characterized by the points 0,0.2,0.4,... and 0,0.02,0.04,... . The range of validity of the linear theory imposes a restriction on the angle ψ: as v increases, the angle ψ becomes smaller.

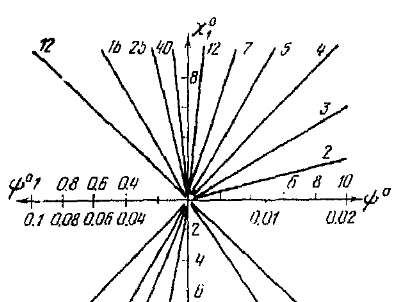

Fig. 2

We used the following numerical data:

$m_1=m_2=14.7$ kg, $m_3=127.5$ kg, $m_4=9.81$ kg, $g=9.81$ msec^{-2}, $R=0.3$ m
$b=0.0724$ m, $c_1=0.05438$ m, $l_1=0.63$ m, $h_1=0.57$ m, $d_1=0.45$ m, $d=0.03$ m
$\varphi=0.023$ rad
$I_1'=I_2'=0.33$ kgm^2, $I_1=I_2=0.667$ kgm^2, $B_1=6.867$ kgm^2, $A_1=11.77$ kgm^2, $D_1=0.196$ kgm^2
$D_2=0.98\cdot10^{-4}$ kgm^2, $B_2=0.0078$ kgm^2, $A_2=0.0098$ kgm^2

For the simplest model of the bicycle,

$$\chi=-\mu_2 g^{-1}\nu_1^{-1}v^2\psi=-\mu_2 c g^{-1}\nu_1^{-1}R_1^{-1}v^2=-[\nu_1+(I_1+I_2)R^{-1}]\nu_1^{-1}g^{-1}R_1^{-1}v^2$$

In the elementary theory, the inertia of the wheels is neglected. Moreover, $\nu_1+(I_1+I_2)R^{-1}\approx\nu_1$ to within about 5%. We then have $\chi_1\approx v^2 g^{-1}R_1^{-1}$. The last formula is given in [9].

3. In the case of a uniformly retarded $(v=v_0-wt,\ w=\text{const}>0)$ controlled motion of the bicycle, and uniformly varying steering angle $(\psi=\beta t,\ \beta=\text{const})$, we have

$$\theta=\frac{\beta}{c}\left[c_1+\left(\frac{v_0}{2}t-\frac{w}{3}t^2\right)\cos\lambda\right]t,\quad k_1=\frac{\beta[c_1+(v_0 t-wt^2)\cos\lambda]}{c(v_0-wt)}$$

$$x=s_1=v_0 t-\frac{w}{2}t^2\quad y=\frac{\beta}{c}\left[\frac{1}{2}v_0 c_1 t^2+\left(\frac{1}{2}v_0^2\cos\lambda-c_1 w\right)\frac{t^3}{3}-\right.$$

$$\left.-\frac{5}{24}v_0 w t^4\cos\lambda+\frac{w^2}{15}t^5\cos\lambda\right]$$

The tilt angle χ is given by the second equation in (1.4):

$$\chi^{\cdot\cdot}-r^2\chi=a_{31}\beta v_0+[(a_{61}-a_{31})w+a_{51}v_0^2-a_{41}]\beta t-2a_{51}\beta v_0 w t^2+a_{51}\beta w^2 t^3$$

$$r^2=a_1/a_0,\quad a_{31}=a_3/a_0,\quad a_{41}=a_4/a_0,\quad a_{51}=a_5/a_0$$

The particular solution of the last equation corresponding to the above forced motion is

$$\chi=\beta r^{-2}\{v_0(4a_{51}r^{-2}w-a_{31})+[a_{41}-a_{51}v_0^2+(a_{31}-a_{61})w-6a_{51}r^{-2}w^2]t+2a_{51}v_0 w t^2-a_{51}w^2 t^3\} \tag{3.1}$$

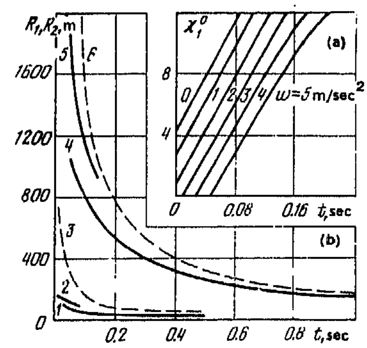

Fig. 3

Figure 3a shows $\chi_1=\chi_1(t)$, $\chi_1=-\chi$ for $\beta=0.1$ rad/sec, v_0 = 15 m/sec. This is a nearly relationship, except that for large values of w the graph of $\chi_1(t)$ is found to curve slightly to the right. The slowing down has practically no effect on the other characteristics of the bicycle. We now note the following point. At the beginning of the motion, $\theta<\psi$. For $w\leq 4$ m/sec^2 and small values of t corresponding to the linear model, the angle θ remains smaller than ψ. When w = 5 m/sec^2, and beginning with t = 0.21 sec, the bicycle is found to turn around $(\theta>\psi)$: for t = 0.21 sec, $\theta=1.22°$, $\psi=1.20°$, $\chi_1=12.5°$; for t = 0.22 sec, $\theta=1.33°$, $\psi=1.26°$, $\chi_1=13.1°$. To establish the relationship between θ and ψ for large values of the time t, we must consider the nonlinear model because the values of the angle χ_1 leave the range of validity of the linear theory.

Let w = 0. We then have $\chi=-\beta a_1^{-1}[a_3 v+(a_5 v^2-a_4)t]$, i.e., the bicycle tilts over uniformly. Calculations show that when at least one of the two quantities β, v increases, the time during which the linear theory remains valid becomes shorter. The radii of curvature

$$R_1=\beta^{-1}(c_1 c^{-1}v^{-1}+c^{-1}t\cos\lambda)^{-1},\quad R_2=\beta^{-1}[(c_1 c^{-1}+\cos\lambda)v^{-1}+c^{-1}t\cos\lambda]^{-1}$$

at the points K_1 and K_2 decrease with increasing angular velocity $\psi^{\cdot}=\beta$ of the front wheel (this is accompanied by a reduction in the interval of time during which the variables remain small). The broken curve in Fig. 3b shows the graph of $R_1(t)$ and the solid curve the graph of $R_2(t)$. Curves 1-3 correspond to $\beta=0.1$ rad/sec and curves 4-6 correspond to $\beta=0.01$ rad/sec. For curve 1 the velocity is v = 5 m/sec, curves 2 and 4 correspond to v = = 15 m/sec, and curve 5 to v = 45 m/sec. Curves 3 and 6 were constructed for 5 m/sec $\leq$ v $\leq$ 15 m/sec and 15 m/sec $\leq$ v $\leq$ 45 m/sec, respectively.

As the velocity v and time t increase, the range of validity of the linear theory is reduced, i.e., there is a simultaneous reduction in both components of the denominators of

R_1 and R_2. However, $c_1c^{-1} \ll \cos\lambda$ (for the above numerical data, $c_1c^{-1}=0.039$, $\cos\lambda=0.906$). This means that R_2 is more sensitive to changes in z than R_1. Examination of the numerical values will show that R_1 is practically independent of v whereas R_1 can be approximately represented by $R_1 \approx c\beta^{-1}t^{-1}\cos^{-1}\lambda$. The results obtained from this approximate expression are slightly too high. The effect of v on R_2 cannot, however, be neglected.

At the beginning of the turning process, when t is small, we have $R_1 \gg R_2$ for small values of v. As the process continues, i.e., as t increases, the values of R_1 and R_2 become equal and the graphs of $R_1=R_1(t)$, $R_2=R_2(t)$ approach one another but $R_1>R_2$ throughout the process.

When w = 1, we obtain a linear dependence of the angle χ on time in (3.1):

$$\chi_1=-\chi=\beta a_1^{-1}[a_3v+(a_5v^2-a_4)t]$$

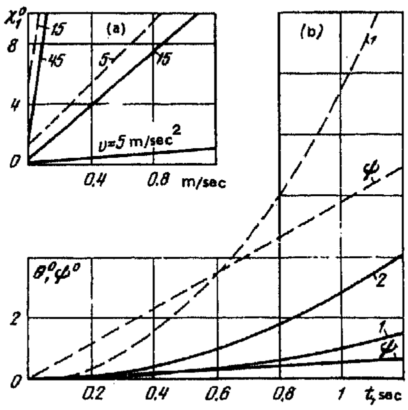

Fig. 4

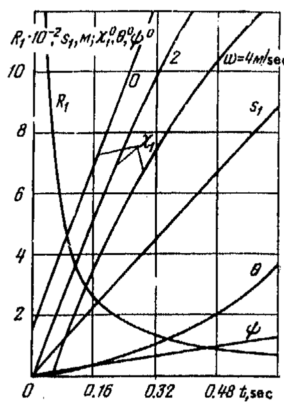

Fig. 5

The solid curves in Fig. 4 correspond to β = 0.01 rad/sec and the broken curves to β = 0.1 rad/sec. The solid straight lines cover the entire quadrant of Fig. 4a, i.e., the velocity v of the bicycle has a considerable effect on the tilt χ_1. An increase in β for v = const is also found to have a sharp effect on the angle χ_1. A simultaneous increase in v and β leads to a considerable increase in χ_1.

The angle θ is given by $\theta=\beta c^{-1}(c_1t+\frac{1}{2}vt^2\cos\lambda)$.

Curve 1 in Fig. 4b is a graph of $\theta=\theta(t)$ for v = 5 m/sec; curve 2 corresponds to v = 15 m/sec. At the beginning of the turning process, $\theta<\psi$, but eventually θ becomes greater than ψ. A certain definite amount of time is necessary before the bicycle can turn, and thereafter the turning process becomes faster than the change in ψ. This is so because, for small t, the angular velocity $\dot{\theta}=\beta c^{-1}(c_1+vt\cos\lambda)$ is smaller than the angular velocity β of the steering system $(c_1/c \ll 1)$. As time increases, the angular velocity $\dot{\theta}$ with which the bicycle turns rapidly exceeds the angular velocity of the steering system. Thus, when v = 5 m/sec, we have $\dot{\theta}/\beta=0.039+3.235t$. When t = 0.2 sec, we have $\dot{\theta}/\beta=0.686<1$. Finally, when t = 0.3 sec, we obtain $\dot{\theta}/\beta=1.01>1$.

The trajectories of the points K_1, K_2 are cubic parabolas whose ordinates are much smaller than the abscissas: $y \ll x$, $y' \ll x'$. This is a consequence of the linearization process: $s_1=vt\approx x$, y being small.

4. Suppose that K_1 moves on a given trajectory (program) and let us refer to the corresponding motion of the bicycle as the programmed motion. It is required to determine the dynamic characteristics of the bicycle.

Let us suppose that the programmed curve is the clothoid $R_1s_1=\gamma$, $\gamma=\text{const}$ and that we have uniform slowing down, i.e., $v=v_0-wt$. We then have

$$R_1=\gamma(v_0t-\tfrac{1}{2}wt^2)^{-1}, \qquad \theta=\tfrac{1}{2}\gamma^{-1}(v_0t-\tfrac{1}{2}wt^2)^2$$

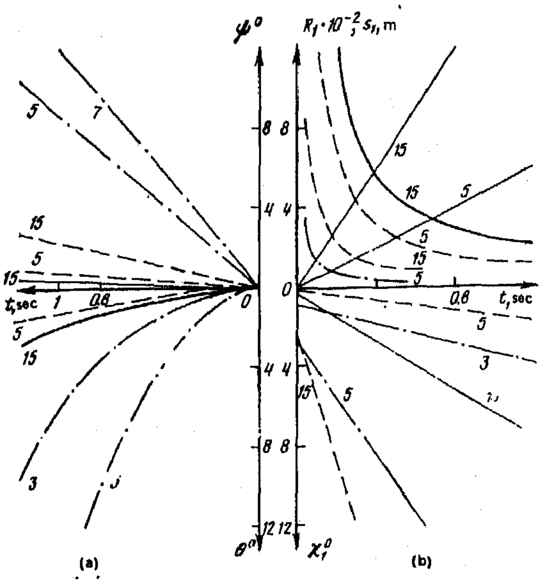

Fig. 6

To ensure that the point K_1 moves on the clothoid from the state $\psi_0=0$, we must take

$$\psi = cc_1\gamma^{-1}\cos^{-2}\lambda\{\exp[-c_1^{-1}(v_0t-{}^1/_2wt^2)\cos\lambda]-1\}+c\gamma^{-1}(v_0t-{}^1/_2wt^2)\cos^{-1}\lambda$$

The angle χ is the solution of the differential equation

$$\begin{aligned}(\chi''-r^2\chi)\gamma c^{-1}\cos\lambda = {}&-a_{21}w+a_{31}(v_0-wt)^2+[a_{61}w-a_{41}+a_{51}(v_0-wt)^2](-c_1/\cos\lambda+\\&+v_0t-{}^1/_2wt^2)+\exp[-c_1^{-1}(v_0t-{}^1/_2wt^2)\cos\lambda]\{a_{21}[w+c_1^{-1}(v_0-wt)^2\cos\lambda]-\\&-a_{31}(v_0-wt)^2+c_1[a_{61}w-a_{41}+a_{51}(v_0-wt)^2]\cos^{-1}\lambda\},\quad a_{21}=a_2/a_0,\quad a_{61}=a_6/a_0\end{aligned} \tag{4.1}$$

Let

$$\begin{aligned}A_0 &= 12a_{51}r^{-6}w^3+(a_{61}+2a_{51}c_1\cos^{-1}\lambda-2a_{31})r^{-4}w^2+5a_{51}r^{-4}v_0^2w+\\&\quad+(a_{21}-a_{41}r^{-2}+a_{61}c_1\cos^{-1}\lambda)r^{-2}w+r^{-2}v_0^2(a_{51}c_1\cos^{-1}\lambda-a_{31})-a_{41}c_1r^{-2}\cos^{-1}\lambda\\A_1 &= -12a_{51}r^{-4}v_0w^2+r^{-2}v_0w(2a_{31}-2a_{51}c_1\cos^{-1}\lambda-a_{61})-a_{51}r^{-2}v_0^3+a_{41}r^{-2}v_0\\A_2 &= 6a_{51}r^{-4}w^3+r^{-2}w^2({}^1/_2a_{61}+a_{51}c_1\cos^{-1}\lambda-a_{31})+{}^5/_2a_{51}r^{-2}v_0^2w-{}^1/_2a_{41}r^{-2}w\\A_3 &= -2a_{51}r^{-2}v_0w^2,\qquad A_4={}^1/_2a_{51}r^{-2}w^3\end{aligned}$$

The particular solution of (4.1) satisfying the initial conditions

$$\chi_0=c\gamma^{-1}A_0\cos^{-1}\lambda,\qquad \chi_0'=c\gamma^{-1}A_1\cos^{-1}\lambda,$$

is

$$\chi=\frac{c}{\gamma\cos\lambda}\left\{\sum_{n=0}^{4}A_nt^n+\frac{\exp(rt)}{2r}\int_0^t[a+b(v_0-wt)^2]\times\right.$$

$$\times\exp\left[\left(-r-\frac{v_0}{c_1}\cos\lambda\right)t+\frac{w\cos\lambda}{2c_1}t^2\right]dt-\frac{\exp(-rt)}{2r}\int_0^t[a+b(v_0-wt)^2]\times$$

$$\left.\times\exp\left[\left(r-\frac{v_0}{c_1}\cos\lambda\right)t+\frac{w\cos\lambda}{2c_1}t^2\right]dt\right\}$$

$$a=a_{21}w+c_1(a_{61}w-a_{41})\cos^{-1}\lambda,\qquad b=a_{21}c_1^{-1}\cos\lambda+a_{51}c_1\cos^{-1}\lambda-a_{31}$$

The curves of Fig. 5 were constructed for $\gamma=600$ m^2, v_0 = 15 m/sec; the graphs of $R_1(t)$, $s_1(t)$, $\theta(t)$, $\psi(t)$ correspond to w = 4 m/sec^2. The acceleration w has a considerable effect on the tilt angle χ; the angles θ, ψ, and also the quantities R_1, s_1, are not very dependent on w and are affected by it only to the extent that, as w increases, the range of validity of the linear theory [0, t] is found to increase.

When w = 0,

$$\chi=c\gamma^{-1}\cos^{-1}\lambda\left\{A_0+A_1t+\frac{a+bv_0^2}{(v_0c_1^{-1}\cos\lambda)^2-r^2}\exp(-v_0c_1^{-1}t\cos\lambda)+\right.$$

$$\left.+\frac{a+bv_0^2}{2r}\left[\frac{\exp(rt)}{r+v_0c_1^{-1}\cos\lambda}+\frac{\exp(-rt)}{r-v_0c_1^{-1}\cos\lambda}\right]\right\}$$

This case is illustrated in Fig. 6 (a and b). The variable parameters are v_0, γ. The solid curves correspond to $\gamma=600$ m^2 and the broken curves to $\gamma=3375$ m^2. The dot-dash curves correspond to $\gamma=50$ m^2. The numbers shown against the curves are the values of z_0 in m/sec. Figure 6a shows graphs of $\psi=\psi(t)$ and $\theta=\theta(t)$. The upper half of Fig. 6b shows graphs of $R_1(t)$ (rays) and $s_1(t)$ (hyperbolic curves). The lower half of Fig. 6b shows graphs of $\chi_1=\chi_1(t)$.

As the parameter γ increases, the curvature of the clothoid decreases. It follows from Fig. 6 that this is accompanied by a reduction in the angles $\psi, \theta, \chi_1=-\chi$ which represent the rotation of the front wheel, the rotation of the bicycle, and the tilt angle.

When γ is kept constant, all three angles increase with increasing initial velocity v_0 of the bicycle, but the tilt angle increases more rapidly than the other angles.

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18 March 1977