An advanced model of bicycle dynamics

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Received 30 June 1989

Abstract. A theoretical model of a moving bicycle is presented for arbitrary bicycle geometries at finite angles. The non-linear equations of motion are derived and solved with the help of a computer. The solutions are tested for energy conservation, and examined with respect to inherent stability. For common bicycles, velocity and lean angle ranges of self-stable motion are predicted.

Zusammenfassung. Ein theoretisches Modell der Dynamik des Fahrrads für beliebige Fahrradgeometrien und endliche Winkel wird entwickelt. Die nichtlinearen Bewegungsgleichungen werden erstellt und mit Computerhilfe gelöst. Die Lösungen werden im Hinblick auf Energieerhaltung und Eigenstabilität untersucht. Für gebräuchliche Fahrräder findet sich ein Geschwindigkeits- und Kippwinkelbereich eigenstabiler Fahrzustände.

1. Introduction

The problem of bicycle stability has been analysed many times at different levels of mathematical skill. At the turn of the century, Whipple [1] and Klein and Sommerfeld [2] obtained self-stabilising characteristics depending on speed: there is a stable region between 4 and $5.5\,\mathrm{m\,s^{-1}}$. The following simplifications and approximations are made: the angles are small, the wheels are equal in diameter, the centre of gravity of the bicycle and rider system is located in the plane of the frame. More recently Jones [3] and Le Hénaff [4] concentrated on geometrical considerations (neglecting dynamical forces on the steering system) and pointed out the crucial role of front fork geometry (especially of the trail length) for the ease of steering—as bicycle builders know by experience. Self-stability was not predicted by their models. Dikarev *et al* [5] investigated the velocity interval for stability of rectilinear motion of an uncontrolled bicycle in relation to steering geometry (head angle and trail length). Computer simulation was first used by Douglas Roland [6] to solve a system of eight differential equations and a huge number of parameters in order to come close to reality. Papadopoulos [7] developed the linearised equations of motion for general bicycle geometry (including dynamical properties of the rider's body) and obtained ranges of stable motion depending on the velocity and parameters of the bicycle.

Our research was aimed at eliminating some of the limitations mentioned above: a general model of the moving bicycle was to be found, with as few simplifications as possible, with any given geometry (also valid for penny-farthings and recumbents), allowing us to determine stable and unstable riding situations, and to investigate the ability of the bicycle to stabilise itself under any given initial conditions—the rider being able to steer with the help of small displacements of the centre of gravity. Nevertheless a few approximations are made: the parts of the bicycle are considered as rigid, there is no friction or slip between wheels and ground, the wheels themselves are infinitely thin, the ground is even, and there is no wind. We concentrated on hands off riding and thus always set the steer torque to zero.

The bicycle has the six degrees of freedom of a rigid body plus three internal ones (steering angle and angular positions of the wheels). It is subjected to two holonomic constraints (both wheels touch the ground plane) which reduce this number to seven, and to four anholonomic constraints (both wheels are rolling) which affect only the degrees of motional freedom reducing them to three. Seven degrees of configurational freedom remain but five of them (the absolute position, the absolute direction of the frame, and the angular positions of the wheels) have no influence on the motion and thus do not have to be considered. We therefore will describe the motion of the bicycle by two plus three variables: the steering angle ϕ, the lean angle κ, the time derivatives $\dot{\phi}$, $\dot{\kappa}$ and the (scalar) velocity v (measured at the rear wheel).

Due to the anholonomic constraints, ϕ, κ and the integral of v are not generalised variables within the meaning of Lagrangian mechanics. It may point to the importance of the anholonomy that obviously the momentum of the bicycle is not conserved although it moves on a translational invariant plane. Thus, we will use Newton's formulation of mechanics to construct the equation of motion.

2. Analysis of bicycle geometry

A precondition of a bicycle model capable of handling even non-standard bicycle geometries within a realistic range of steering and lean angles is a comprehensive analysis of bicycle geometry. As Jones [3] commented bicycle geometry is a 'remarkably tricky little problem'. Nevertheless, the complexity of this geometrical problem can be reduced when selecting an appropriate system of coordinates.

Figure 1. Definition of frame parameters and angles.

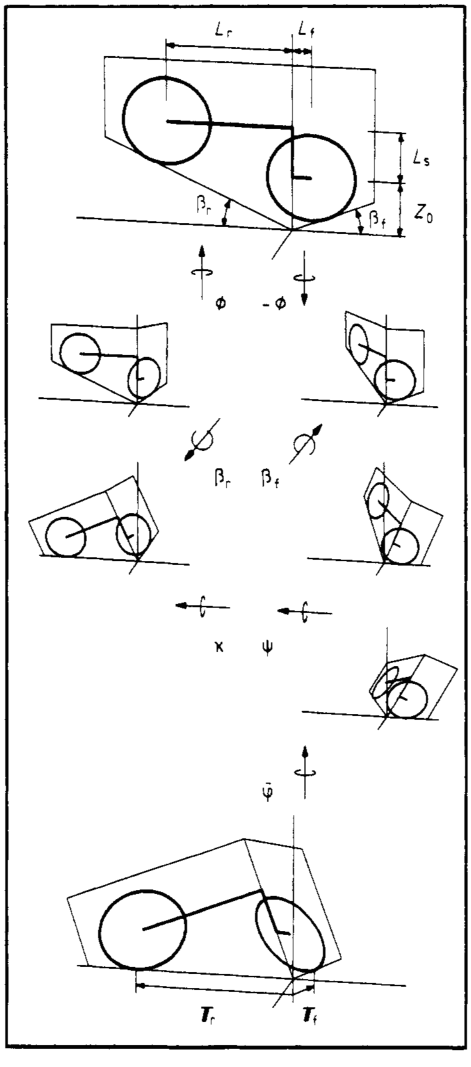

Figure 1 (bottom) shows the bicycle standing on the ground plane illustrating that both planes within which the wheels are located intersect along the steering axis. For this reason the point of interception of the steering axis with the ground plane is selected as the origin of an appropriate system of coordinates suitable for any plane frame geometry. Cutting off the wheel planes along the lines of their intersection with the ground they can be unfolded into a plane as shown in figure 1 (top). We will refer to this position as the 'plane reference position' where the steering axis coincides with the vertical axis of an orthogonal system of coordinates.

Five geometric parameters are used to describe the bicycle: the three lengths L_r, L_f, L_s which characterise the frame and the radii R_f, R_r of front and rear wheels, respectively. These five parameters plus the two variables—lean angle κ and steering angle ϕ—determine the three-dimensional position of the bicycle standing on the ground. Our task is now to determine the ground contact points of the wheels, the track of the bicycle's motion and the positions of the bicycle components from these variables.

Therefore we define three auxiliary variables β_f, β_r and Z_0 as can be seen in the plane reference position (figure 1, top). In the normal position of the bicycle β_f and β_r can be obtained as follows: we construct auxiliary lines in the planes of the front and rear wheels perpendicular to the respective intersections with the ground plane. β_f and β_r are the angles between these auxiliary lines and the steering axis. The distance from the point on the steering axis which is nearest to the centre of the front wheel to the origin is called Z_0.

Geometrical relations for these quantities are obtained by transforming the bicycle from its plane reference position into the ordinary three-dimensional position. These transformations consist of two alternative sets of rotations using either the angles ϕ, β_r, κ or $-\phi$, β_f, ψ, $\tilde{\varphi}$ (where ψ is the lean angle of the front wheel and $\tilde{\varphi}$ the projection of the steering angle on the ground plane). The above rotations apply to all bicycle components except for the first rotation of the respective set ($\pm\phi$, corresponding to the steering action) which turns only the front or the rear components about the steering axis.

In order to calculate the four unknown angles, we make use of the equivalence of the two sets of rotations. The identity of the respective matrices in the coordinates of figure 1 leads to three independent equations:

$$\tan\beta_f = \tan\beta_r\cos\phi + \tan\kappa\sin\phi/\cos\beta_r \quad (1)$$

$$\sin\psi = \cos\phi\sin\kappa - \sin\phi\cos\kappa\sin\beta_r \quad (2)$$

$$\sin\tilde{\varphi} = \sin\phi\cos\beta_f/\cos\kappa. \quad (3)$$

These equations indicate that the determination of β_r plays a key role for the computation of the variables β_f, ψ, $\tilde{\varphi}$. The fourth equation to complete the system is a geometrical relation obtained from the plane reference position:

$$L_r \tan\beta_r + \frac{R_r}{\cos\beta_r} = -L_f \tan\beta_f + \frac{R_f}{\cos\beta_f} + L_s. \tag{4}$$

Since ψ and $\tilde{\varphi}$ do not appear in equations (1) and (4), the problem is reduced to the simultaneous solution of these two equations. An analytical approach turned out to be discouraging, thus the respective values of β_r and β_f are calculated numerically.

To develop the equations of motion, a system of coordinates is used moving along with the bicycle with its origin fixed at the point of interception of steering axis and ground plane. The accompanying unit vectors point into the direction of the hub of the rear wheel ($\boldsymbol{e}_N$), its rolling direction ($\boldsymbol{e}_R$) and from the wheel's ground contact point to its centre ($\boldsymbol{e}_A$).

Defining the distance of the origin to the respective ground contact points of the wheels as $\boldsymbol{T}_f = -T_f\boldsymbol{e}_R^f$ and $\boldsymbol{T}_r = -T_r\boldsymbol{e}_R$ (see figure 1 (bottom)) where $\boldsymbol{e}_R^f$ points into the rolling direction of the front wheel, geometrical considerations yield

$$T_f = -L_f/\cos\beta_f + R_f \tan\beta_f \tag{5}$$

$$T_r = L_r/\cos\beta_r + R_r \tan\beta_r. \tag{6}$$

(T_f is usually called 'trail', provided that $\phi = 0$.) The unit vector $\boldsymbol{e}_S$ points from the origin to the upward direction of the steering axis. With $\boldsymbol{Z}_0 = Z_0\boldsymbol{e}_S$ the magnitude of $\boldsymbol{Z}_0$ is

$$Z_0 = L_r \tan\beta_r + R_r/\cos\beta_r - L_s. \tag{7}$$

3. Dynamics

In our model, the bicycle consists of five main components: frame, rear wheel, rider, handlebars/fork and front wheel. (The last two components we will denote as the steering system.) We assume that these components are absolutely rigid and assign to each of them a momentum $\boldsymbol{p}_c$ and (in principle) an angular momentum $\boldsymbol{L}_c$ for the rotation about its centre of mass. The theory of bicycle geometry discussed above enables us to calculate the positions, velocities and accelerations of the five components depending on the angles ϕ, κ, their time derivatives and the speed of the bicycle v relative to a system of coordinates that moves along with the bicycle. ($\boldsymbol{v}$ always points in the direction $\boldsymbol{e}_R$.)

The coordinates of a point fixed to the bicycle are

$$\boldsymbol{x} = \mathbf{D}(\beta_r, \phi)\underbrace{(\boldsymbol{x}_0 + \boldsymbol{Z}_0)}_{\boldsymbol{x}_1} \tag{8}$$

where $\boldsymbol{x}_0$ are the coordinates of the point with respect to the plane reference position (they are constant except for the rider who is allowed to move relative to the bicycle frame) and $\mathbf{D}$ is the matrix representation of the rotation described above. κ-dependent terms do not appear because the system of coordinates is turned together with the bicycle, the ϕ-dependent part of $\mathbf{D}$ is used only for the steering system. In the time derivative of equation (8), we have to insert additional terms due to the motion of the system of coordinates:

$$\dot{\boldsymbol{x}} = \dot{\mathbf{D}}\boldsymbol{x}_1 + \mathbf{D}\dot{\boldsymbol{x}}_1 + \mathbf{A}\boldsymbol{x} + \boldsymbol{v} - \mathbf{A}\boldsymbol{T}_r \tag{9}$$

$$\ddot{\boldsymbol{x}} = \ddot{\mathbf{D}}\boldsymbol{x}_1 + \mathbf{D}\ddot{\boldsymbol{x}}_1 + 2\dot{\mathbf{D}}\dot{\boldsymbol{x}}_1 + (\dot{\mathbf{A}} - \mathbf{A}^2)(\boldsymbol{x} - \boldsymbol{T}_r) + \dot{\boldsymbol{v}} + \mathbf{A}(2\dot{\boldsymbol{x}} - \boldsymbol{v}). \tag{10}$$

$\mathbf{A}$ is a matrix constructed from the time derivatives of the unit vectors:

$$\mathbf{A} = \begin{pmatrix} 0 & -\Omega\cos\kappa & \dot{\kappa} \\ \Omega\cos\kappa & 0 & \Omega\sin\kappa \\ -\dot{\kappa} & -\Omega\sin\kappa & 0 \end{pmatrix}. \tag{11}$$

Here, Ω is the angular velocity of the motion about the instantaneous centre of rotation. It can be calculated together with the front wheel speed v_f from the constraint of rolling of the two wheels:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{T}_f - \boldsymbol{T}_r) = v_f\boldsymbol{e}_R^f - v\boldsymbol{e}_R = \mathbf{A}(\boldsymbol{T}_f - \boldsymbol{T}_r) - \dot{T}_f\boldsymbol{e}_R^f + \dot{T}_r\boldsymbol{e}_R \tag{12}$$

leading to

$$\Omega = \frac{(v + \dot{T}_r)\sin\tilde{\varphi} + T_f\dot{\tilde{\varphi}}}{T_r\cos\tilde{\varphi} - T_f} \tag{13}$$

$$v_f = (v + \dot{T}_r)\cos\tilde{\varphi} - \dot{T}_f + T_r\Omega\sin\tilde{\varphi}. \tag{14}$$

Applying the formalism of equations (8)–(10) to the centres of mass $\boldsymbol{x}_c$ of the bicycle components allows the calculation of the momenta $\boldsymbol{p}_c = m_c\dot{\boldsymbol{x}}_c$. Only the most important angular momenta are taken into account, those of the rear and front wheel (in hub direction) and of the steering system:

$$\boldsymbol{L}_f = -\Theta_f(v_f/R_f + (\Omega + \dot{\varphi})\sin\psi)\boldsymbol{e}_N^f \tag{15}$$

$$\boldsymbol{L}_r = -\Theta_r(v/R_r + \Omega\sin\kappa)\boldsymbol{e}_N \tag{16}$$

$$\boldsymbol{L}_S = \Theta_S(\dot{\phi} + \Omega\cos\kappa\cos\beta_r - \dot{\kappa}\sin\beta_r)\boldsymbol{e}_S \tag{17}$$

where Θ is the respective moment of inertia and $\boldsymbol{e}_N^f$ is a unit vector pointing in the direction of the hub of the front wheel.

With these dynamical quantities, we are now able to construct the bicycle's equation of motion. Three forces act on the bicycle from outside: the forces of the ground on the rear and front wheels $\boldsymbol{F}_r$, $\boldsymbol{F}_f$ and the weight $\boldsymbol{G} = \Sigma_c\boldsymbol{G}_c$. Thus, for the whole bicycle, the equations of translation and rotation are

$$\sum_c(\dot{\boldsymbol{p}}_c - \boldsymbol{G}_c) = \boldsymbol{F}_f + \boldsymbol{F}_r \tag{18}$$

$$\sum_c(\dot{\boldsymbol{L}}_c + \boldsymbol{x}_c \times (\boldsymbol{p}_c - \boldsymbol{G}_c)) = \boldsymbol{T}_f \times \boldsymbol{F}_f + \boldsymbol{T}_r \times \boldsymbol{F}_r \tag{19}$$

where the sum Σ_c ranges over all five bicycle components. For the internal degrees of freedom (rotation of the wheels and steering), only scalar equations of motion remain:

$$e_N \dot{L}_r = R_r e_R F_r \quad (20)$$

$$e_N^f \dot{L}_f = R_f e_R^f F_f \quad (21)$$

$$e_S \sum_{fc} (\dot{L}_c + x_c \times (\dot{p}_c - G_c)) = e_S T_f \times F_f + M_S. \quad (22)$$

Here, M_S is the steer torque and the sum Σ_{fc} ranges over the steering system. (The sum index rc will represent the other bicycle components.)

From the nine scalar equations (18)–(22), we eliminate the forces F_r, F_f by tedious but straightforward calculation resulting in the final equation of motion as a three component vector equation. Using the abbreviations $w_1 = (\sin\tilde{\varphi}/T_r \cos\beta_r)e_z$ with e_z perpendicular to the ground, $w_2 = -\sin\kappa \sin\tilde{\varphi} e_R - \cos\tilde{\varphi} e_A$, $s = -\sin\kappa\sin\tilde{\varphi}$ and $K = T_f - T_r$, we obtain the equation of motion:

$$\sum_c [(\dot{L}_c + (x_c - T_f) \times (\dot{p}_c + G_c))s + e_R^f \dot{p}_c K \times e_A] + \left\{ \sum_{rc} [(e_S \times x_c)(\dot{p}_c + G_c) + e_S \dot{L}_c] + M_S \right\} K \times w_1 + \frac{e_N \dot{L}_r}{R_r} K \times w_2 - \frac{e_N^f \dot{L}_f}{R_f} K \times e_A = 0. \quad (23)$$

4. Numerical solution

The equation of motion derived above is implicit, i.e. it is not solved for the second derivatives $\ddot{\phi}$, $\ddot{\kappa}$, $\dot{v}$ of the dynamical variables, thus it cannot be solved by numerical methods directly. From the structure of Newton's law of motion and the rules of differentiation, we know that equation (23) depends on the second derivatives linearly. So, if the variables and their first derivatives are known, their calculation reduces to the determination of the zero of a linear function which can be solved exactly.

Having made the equation of motion explicit this way, we implemented it on a computer and solved it numerically by a Runge–Kutta procedure assuming an absolutely passive rider fixed to the bicycle. This turned out to be quite easy in a range of $|\kappa| < 45°$. As the positions, momenta and angular momenta of the bicycle components are known, the total energy

$$E = \sum_c \frac{p_c^2}{2m_c} + \frac{L_c^2}{2\Theta_c} + G_c x_c \quad (24)$$

can be calculated very easily. In the numerical simulation, we were able to approach an accuracy of $\Delta E/E < 10^{-8}$ per second.

The system exhibits a very manifold dynamical behaviour dependent on the energy; in particular a region of stable stationary motion exists with a very large basin of attraction. Also from initial conditions very far away, the system tends to the stationary values of the coordinates. We remark that this is due to the anholonomic constraints; Liouville's theorem prevents Hamiltonian systems from this kind of behaviour. We will restrain ourselves here to the discussion of the linear stability of stationary motion.

We transform our system to first order and omit the speed v, which can be calculated from the other variables by means of energy conservation. In a vector notation $\alpha = (\phi, \kappa, \dot{\phi}, \dot{\kappa})$, our dynamical system is given by

$$\dot{\alpha} = F(\alpha) \quad (25)$$

with the condition of stationarity

$$F(\alpha_0) = 0. \quad (26)$$

A fixed point α_0 of this system corresponds to stationary circular motion of the bicycle; to allow arbitrary radii, we introduce as an additional parameter a constant lateral displacement w of the rider's centre of mass (of the magnitude of several centimeters). For given speed v and displacement w, we found several fixed points with different ϕ and κ; the only possible way to show the important ones in one plot was to use κ as an independent variable and to calculate the corresponding w. Figure 2(a,b) shows ϕ and w for given v and κ for a typical roadster bicycle with a wheelbase of 111 cm, a trail of 7 cm and a rider of 70 kg.

Exceeding a certain speed limit (here $\simeq 7\,\mathrm{m\,s^{-1}}$) figure 2($b$) shows only one fixed point for given w. Below that limit two fixed points occur (for positive w), for reasons of symmetry a third fixed point is located at negative κ.

In the vicinity of the fixed points, we linearised F (by numerical differentiation) and diagonalised the matrix of the linear mapping. Figure 2(c) shows the largest eigenvalue ε_{max} of the linearised dynamical system. In the region enclosed by the zero contour (self-stability region), all eigenvalues are negative and the system is self-stable in the sense that in a finite region around α_0, α approaches α_0 exponentially. Where ε_{max} is positive, the fixed point is unstable in the sense that a perturbation in the direction of the corresponding eigenvector grows $\sim \exp(\varepsilon_{max} t)$. Thus, $1/\varepsilon_{max}$ is a measure for the time within which the rider would have to react to a perturbation of the bicycle's motion.

Above the stable region, there is one real positive eigenvalue with a corresponding eigenvector with a large κ component indicating an unstable motion by turning over with too slow a reaction of the steering system. Below the stable region, the real part of two complex conjugate eigenvalues becomes positive, indicating a locally unstable oscillation. Within the stability region, the amplitudes of ϕ and κ in that oscillation are nearly equal while the phaseshift is $\simeq \pi$ (when the bicycle is leaned to one side the front wheel is turned to the same side); at lower speed, the ϕ oscillations become larger while the phaseshift decreases: ϕ approaches its maximum later than κ and

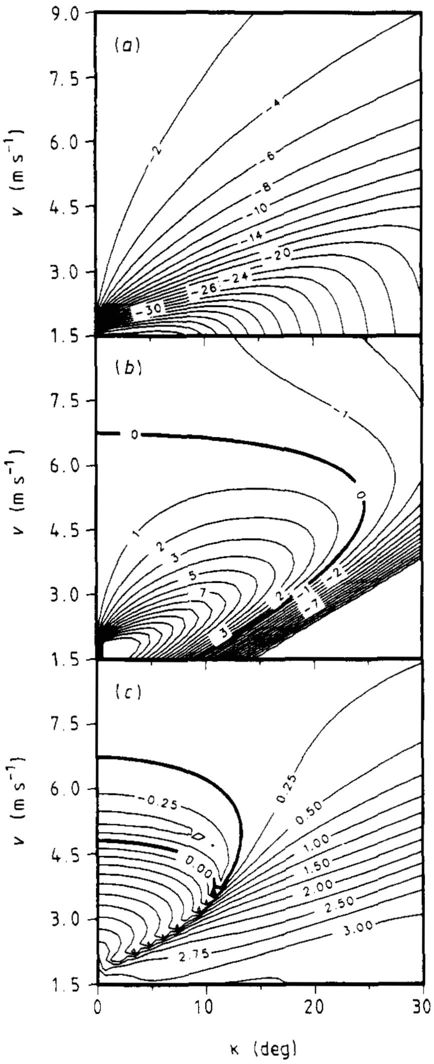

Figure 2. Parameters of stationary motion as functions of lean angle and velocity: (a) Steering angle ϕ (deg); (b) lateral displacement w (cm); (c) largest eigenvalue ε_{max} (s^{-1}).

thus 'comes too late'. Just below the lower limit of the stability region, the instability for this oscillation is stopped by non-linear effects at a finite amplitude (Hopf bifurcation).

The shape of the stability region of figure 2(c) is typical for all kinds of bicycles we investigated with the discussed method. The gyroscopic effects of the rotation of the front wheel are essential for inherent stability. Figure 3 shows the dependence of the limits of the stability region at $\kappa = 0$ on the moment of inertia Θ of both wheels. The absolute size of the region grows, while the speed where it occurs approaches infinity, as Θ tends to zero. (Note that the lower limit

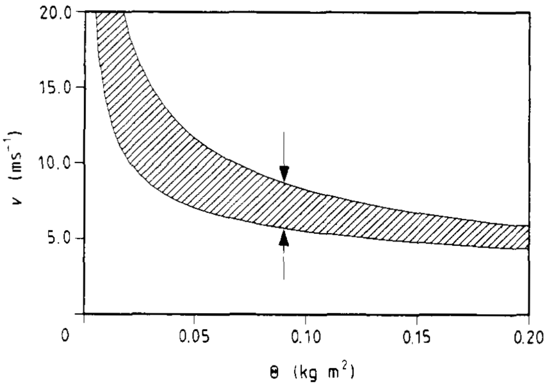

Figure 3. Self-stability region (shaded) as a function of the moment of inertia of both wheels. (Arrows indicate the values from figure 2.)

tends to a large but finite value if the rear wheel keeps its moment of inertia.) The dependence on the trail as an important parameter in frame construction (at constant head angle) is shown in figure 4. The stability region grows linearly with the trail while it tends to large velocities. Other quantities we found to be important for the self-stability are the following.

- Shifting the position of the rider forward causes a similar effect to that caused by lengthening the trail.
- A smaller head angle (at constant trail) raises slightly the upper limit of the stability region.
- The lower limit is raised by an increase in moment of inertia of the steering system while the upper limit remains constant.
- A viscous damping of the steering system normally has only little effect, but for a recumbent (or 'easy racer' model) which is stable only at high speed, it causes a decrease of the lower limit of about 10%.

Figure 4. Self-stability region (shaded) as a function of the trail. (Arrows as for figure 3.)

5. Conclusion

Using an absolutely rigid uncontrolled bicycle model, for common geometries we found a limited region of inherent stability which is essentially dependent on the gyroscopic forces on the front wheel; other parameters only influence the limits of this region. This result is in accordance with former works [1], [2] for the special case of rectilinear motion, i.e. $\phi = \kappa = 0$. Even without a rider, most of the bicycles are able to move stably at moderate velocities which can easily be checked experimentally.

Nevertheless, the question of what role the inherent stability plays for a bicycle guided by a human rider remains open. As figure 2 shows, in the stable region the stationary state is extremely sensitive to the lateral displacement w and a rider would hardly be able to fix his centre of mass with the required accuracy. Also, experimental tests [3] indicate that bicycles with the angular momentum of the front wheel totally compensated can be ridden 'hands-off' by a very skilled rider; obviously the loss of inherent stability can be overcome by special training of the rider's balancing abilities. Therefore it would be an interesting task to investigate whether the bicycle can be controlled 'hands-off' by lateral motion of the rider's body in the vicinity of a fixed point.

Acknowledgment

We would like to thank R Pivit for stimulating discussions and practical hints. Also we acknowledge helpful comments from Y Le Hénaff and J Papadopoulos. Numerical calculations were performed at the computing centre of our university.

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