CALSPAN CORPORATION
Buffalo, New York 14221

# BICYCLE DYNAMICS - SIMPLIFIED DYNAMIC STABILITY ANALYSES

Task Order No. 8

Calspan Report No. ZN-5921-V-2

Prepared for:

Schwinn Bicycle Co.
Chicago, Illinois

September 1976

## FOREWORD

This report covers the work performed by Calspan Corporation under Purchase Order No. 100789 from the Schwinn Bicycle Co. on Task 8: Simplified Dynamic Analyses.

The period of performance was from 19 April 1976 to 19 July 1976. The project engineer was Roy S. Rice, Principal Engineer in the Transportation Research Department. The technical monitor for Schwinn Bicycle Co. was Eugene Symski.

The author gratefully acknowledges the contributions of Dennis T. Kunkel of the Transportation Research Department on the numerical computations and for the many helpful discussions with him on the approach and analysis.

This report has been reviewed and is approved by:

Edwin A. Kidd

Edwin A. Kidd, Head
Transportation Research Department

TABLE OF CONTENTS

## LIST OF FIGURES



## LIST OF TABLES

## 1. INTRODUCTION

A primary goal of the analyses of bicycle dynamics that Calspan Corporation has performed for the Schwinn Bicycle Co. over the past several years is the definition of stability indices on which to base evaluations of bicycle stability and control characteristics. Specifically, the objective has been to develop measurements of performance, in terms of the bicycle design parameters and operational conditions, which could be used as criteria for identifying acceptable bicycle configurations. In earlier studies (References 1 and 2), generalized constant coefficient models of bicycle steady-state response characteristics were developed and applied to this problem. Several performance parameters (or stability indices) were derived in this work -- the concept of inversion speed, the identification of minimum theoretical free control speed, and the definition of a number of control sensitivity terms. In the current work, the steady-state equations have been expanded to include the dynamic transient terms (albeit still in a simplified linear form) in four primary degrees of freedom -- side force, yaw moment, roll moment, and steer torque.

The simultaneous solution of these equations (which can be performed in a variety of ways) leads to the derivation of the characteristic expression for the response of the vehicle. This expression, which contains all of the static and dynamic terms related to the motion variable of interest, defines the stability of the vehicle (when the coefficients are evaluated for explicit designs and operating conditions). Using these results, critical design factors can be identified which will supplement the steady-state stability indices previously developed for application to performance evaluations.

This report contains a technical discussion (Section 2) which describes the development of the equations of motion and the characteristic stability expression. The elements of this expression are analyzed in terms of the design variables of significance and numerical computations illustrating the applicability of the method are given. Some general conclusions about bicycle design (in the framework of these results) are given in Section 3 and recommendations for formalizing use of the technique by establishing limits based on bicycle tests are given in Section 4. A list of references is contained in Section 5. Two appendices are included -- Appendix A gives a listing and definitions of the symbols used in the text, and Appendix B contains a brief discussion of two examples of the solution of the characteristic stability equation.

## 2. TECHNICAL DISCUSSION

The development and application of the simplified dynamic analysis of bicycle stability and control is presented in this section. The various subsections cover different aspects of the study and these are supported, where suitable, by appendices which contain additional mathematical detail.

### 2.1 Equations of Motion

This analysis of bicycle stability is based on a linear constant-coefficient model of the bicycle-rider system drawing in part on other models as described in References 1 to 4. The equations of motion which describe the bicycle's response in the four primary degrees of freedom are presented and discussed briefly in the following paragraphs. These equations (and the associated coefficients for each of the variables) are subject to the following assumptions:

1. The small angle approximations are employed -- $\sin X = \tan X = X$; $\cos X = 1$. This limits strict applicability to small motions about the upright, straightahead condition but general trends in stability can still be adequately evaluated well beyond this range.

2. Tire cornering performance characteristics are represented as linear functions of slip angle. Tire camber thrust is assumed to be negligible* and all other tire forces and moments are neglected.

3. Products of inertia are neglected. The vertical plane of the rear frame is assumed to be a plane of symmetry.

*This assumption is supported by the results of tire tests performed at Calspan which show the ratio of camber thrust coefficient to cornering stiffness coefficient to be about .01 (Reference 5).

4. Tire forces build up instantaneously with slip angle; that is, there is no time lag (tire relaxation length effect) in tire force development. Although this factor is known to be significant in two-wheel vehicle dynamics (References 3 and 6, for example), values for bicycle tires are not known and, in any case, are thought to be negligible because of the stiffness of a bicycle tire carcass when properly inflated.

5. Velocity is treated as a constant. The equations are therefore not applicable to braking or accelerating situations and the load distribution is held at the at-rest values.

6. The rider is assumed to be rigidly attached to the vehicle except for upper body rotational motion about an axis parallel to the system longitudinal (X) axis.

The symbols used in the equations of motion given below are listed and defined in Appendix A. Most of the lineal and angular dimensions are also shown in Figure 1. For convenience in writing the equations, the symbol A is used to represent the interaction of several front fork design features.

$$A = Z_F t - M_F g f = \frac{M g b}{\ell} t - W_F f$$

This expression defines the manner in which the trail ($t$) and mass offset ($f$), acting on the front tire vertical load ($Z_F$) and the steering assembly mass ($M_F$), produce roll and steer torques with roll and steer displacements. These torques are independent of tire characteristics. Note that $Z_F$ is simply the portion of the total system (bicycle and rider) weight which acts at the front wheel ($\frac{M g b}{\ell}$). For reasonable bicycle designs, $Z_F t$ is much larger than $M_F g f$; it is clear that if the c.g. is shifted toward the rear and the trail is made small, this disproportion may no longer hold.

Side Force Equation

The dynamic side force equation relates the bicycle lateral motions

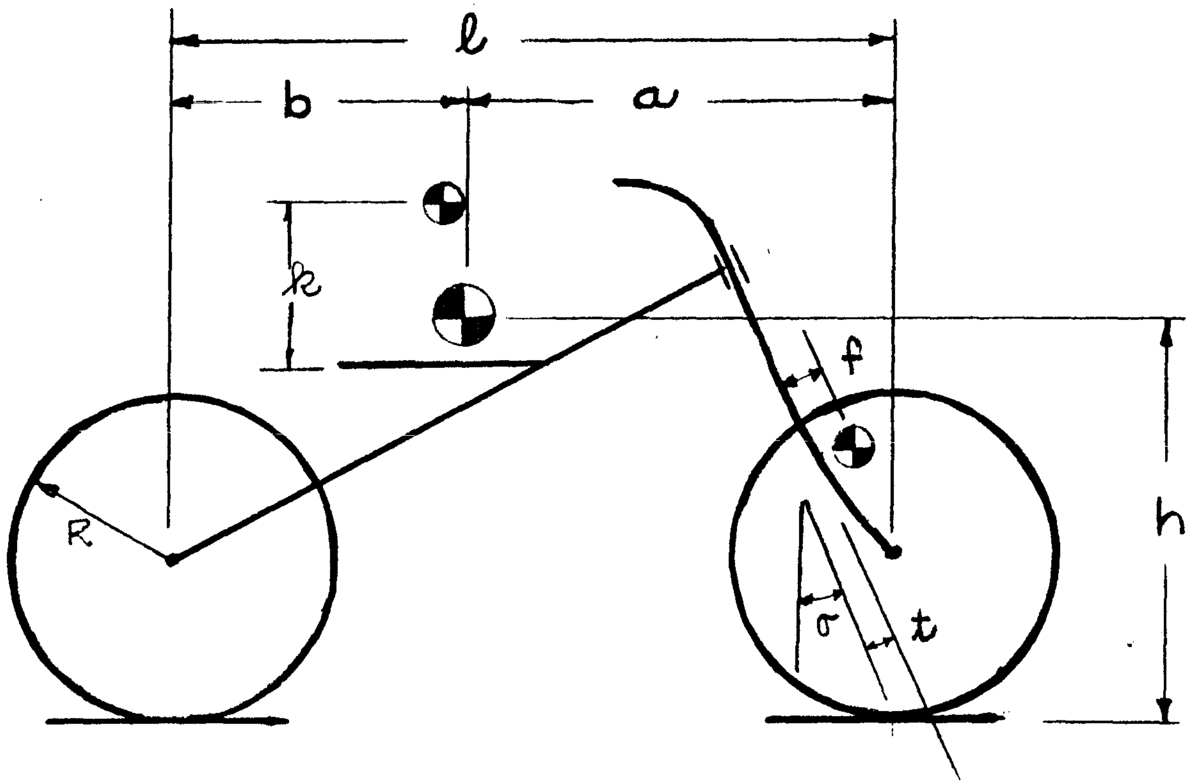


FIGURE 1

CHARACTERISTIC DIMENSIONS OF BICYCLE

to the forces generated by the tires. In terms of the primary motion variables--

$$V(MVs - C_{\alpha F} - C_{\alpha R})\beta + (MV^2 - aC_{\alpha F} + bC_{\alpha R})r$$
$$+ MhV\varphi s^2 + (M_F f V s^2 + C_{\alpha F} t s$$
$$+ C_{\alpha F} V \cos\sigma)\delta = 0$$

The above expression can be developed from the tire force expressions--

$$F_{yF} = \alpha_F C_{\alpha F} = C_{\alpha F}\left[\beta + \frac{ar}{V} - \delta\left(\frac{t}{V}s + \cos\sigma\right)\right]$$

$$F_{yR} = \alpha_R C_{\alpha R} = C_{\alpha R}\left[\beta - \frac{br}{V}\right]$$

Note that a damping term ($\frac{t}{V}$) proportional to the rate of change of the steer angle is included and that camber thrust terms (which had been included in the steady-state analysis - Reference 2) have been neglected. These same tire force expressions are also utilized in the other motion equations which follow.

Yaw Moment Equation

The yaw moment equation relates the bicycle motions about a vertical axis to the moments in a horizontal plane generated by the tires about the c.g. Thus--

$$V(-aC_{\alpha F} + bC_{\alpha R})\beta + (VI_z s - a^2 C_{\alpha F} - b^2 C_{\alpha R})r$$
$$- \frac{i_T}{R}V^2\varphi s + \left(I_3 V\cos\sigma\, s^2 - \frac{i_F}{R}V^2 \sin\sigma\, s\right.$$
$$\left. + atC_{\alpha F}s + aVC_{\alpha F}\cos\sigma\right)\delta = 0$$

The coefficient on $\varphi$ indicates the coupling of the gyroscopic moment due to wheel rotation into this degree of freedom. Note that there are two damping terms on steer angle, one of which is also a gyroscopic effect.

Roll Moment Equation

In this equation, the upsetting moments due to rotational motions about the X-axis are equated to restoring moments due to curvilinear motion and steer angle effects.

$$Mh V \beta s + \left(MhV + \frac{i_T}{R} V\right) r + \left[\left(I_x + Mh^2\right) s^2 - Mhg\right] \varphi + \left(I_z \sin\sigma + \frac{i_F}{R} V \cos\sigma + A\right) \delta = 0$$

In this form, the equation implies that the rider stays in plane with the bicycle frame. Rider lean angle control can be included by replacing the zero value on the right hand side of the equation with a term, $M_R g k \varphi_R$, which represents the roll moment produced by the rider when a relative angular displacement ($\varphi_R$) is developed between the bicycle frame and the rider's upper body. In this model of the rider-bicycle system, this effect does not influence the formulation of the characteristic expression. It might also be mentioned at this point, that for reasonable designs, $Mh \gg \frac{i_T}{R}$ (by about a factor of 30-50) and that $\frac{i_T}{R}$ may therefore be neglected in this term.

Steer Torque Equation

This equation describes the relationship of moments about the steering axis to the requirements for control torque application by the rider. (If the value of T = 0, the equation reduces to the free-control mode of operation.) In terms of the four primary motion variables--

$$\left(M_F f V^2 s + t V C_{\alpha F}\right) \beta + \left[I_z V \cos\sigma\, s^2 + \left(M_F f + \frac{i_F}{R} \sin\sigma\right) V^2 + a t C_{\alpha F}\right] r + \left[I_z V \sin\sigma - \frac{i_F}{R} V^2 \cos\sigma + AV\right] \varphi + \left[I_z V s^2 + \left(KV - C_{\alpha F} t^2\right) s + AV \sin\sigma - C_{\alpha F} t V \cos\sigma\right] \delta = T$$

The steer torque expression is clearly the most complicated of the four equations. It indicates the couplings of the other motion variables into the steer degree of freedom and contains all of the important front end geometry characteristics. For completeness, a steer damping coefficient, K, is included in addition to the trail effect.

## 2.2 Characteristic Expression

The equations have been combined in matrix form in Figure 2. The value of the determinant was then derived in literal form to define the characteristic stability expression as described below.

The solution of the 4x4 determinant yields a sixth order polynomial which is the characteristic expression for the bicycle response to either a steering torque or rider lean angle imput. This expression may then be used to examine the stability of the vehicle (as a function of speed) by insertion of the appropriate values of the design parameters. Further simplification may be justified at this stage by comparing the relative values of the elements in each of the coefficients and discarding terms which are small. Care must be taken in assessing the velocity-dependent terms which may be discarded.

Beyond those assumptions listed earlier, several further simplifications of the general analysis were made once the model was developed in order to arrive at expressions which could be understood in basic design terms. These simplifications included:

1. Neglect of the steer damping term. Although there is very probably a damping effect present, it is not clear that it can be adequately represented as simple viscous friction (i.e., proportional to steering rate). Stiction and coulomb friction effects may also be present (and even dominate). These cannot be effectively treated with the linear model used here.

2. Neglect of the gyroscopic term in the coefficient of yaw rate in

| | | | |
|---|---|---|---|
| $MV^2 s - V(C_{\alpha F} + C_{\alpha R})$ A | $MV^2 - aC_{\alpha F} + bC_{\alpha R}$ B | $MhVs^2$ C | $M_F f V s^2 + t C_{\alpha F} s + V C_{\alpha F} \cos\sigma$ D |
| $V(-aC_{\alpha F} + bC_{\alpha R})$ E | $VI_z s - (a^2 C_{\alpha F} + b^2 C_{\alpha R})$ F | $-\frac{V^2 i_T}{R} s$ G | $VI_z \cos\sigma\, s^2 + \left(atC_{\alpha F} - \frac{V^2 i_F}{R}\sin\sigma\right)s + VaC_{\alpha F}\cos\sigma$ H |
| $MhVs$ J | $MhV$ K | $(I_x + Mh^2)s^2 - Mhg$ L | $I_z \sin\sigma\, s^2 + \frac{V i_F}{R}\cos\sigma\, s + A$ M |
| $M_F f V^2 s + VtC_{\alpha F}$ N | $VI_z \cos\sigma\, s + V^2\left(\frac{i_F}{R}\sin\sigma + M_F f\right) + atC_{\alpha F}$ P | $VI_z \sin\sigma\, s^2 - \frac{V^2 i_F}{R}\cos\sigma\, s + VA$ Q | $VI_z s^2 + (KV - C_{\alpha F} t^2)s + (AV\sin\sigma - VtC_{\alpha F}\cos\sigma)$ R |

Figure 2: BICYCLE CHARACTERISTIC EQUATION DETERMINANT

the roll moment equation (i.e., $\frac{I_T}{R}V$). In all reasonable bicycle designs, the value for this factor is of the order of only a few percent of the dominant MhV term.

3. Omission of many inertial coupling terms. Terms involving $M_F$ and $I_\mathfrak{z}$ are small with respect to terms involving M, $I_x$, and $I_z$ and they have been neglected in the final formulation to reduce the complexity of the coefficients.

4. Neglect of terms involving $(aC_{\alpha F} - bC_{\alpha R})$. In effect, this assumes that the bicycle is neutral steering (which is the case for $aC_{\alpha F} = bC_{\alpha R}$). Most bicycles tend to be slightly oversteer $(aC_{\alpha F} > bC_{\alpha R})$ but previous analyses have shown that the effect is not very significant at normal operating speeds. This assumption will introduce small errors in numerical values but will not affect the trends which are demonstrated.

The general form of the expression is:

$$\Delta_T(s) = A_\Delta s^6 + B_\Delta s^5 + C_\Delta s^4 + D_\Delta s^3 + E_\Delta s^2 + F_\Delta s + G_\Delta$$

where the symbol $\Delta$ is used to identify the characteristic expression, $s$ is the derivative operator (meaning $\frac{d}{dt}$), and --

$$A_\Delta = M I_x I_z I_\mathfrak{z} V^2$$

$$B_\Delta = V\left[-M I_x I_z C_{\alpha F} t^2 - M I_x I_\mathfrak{z}(a^2 C_{\alpha F} + b^2 C_{\alpha R}) - I_z I_\mathfrak{z}(I_x + Mh^2)(C_{\alpha F} + C_{\alpha R})\right]$$

$$C_\Delta = C_{\alpha F} C_{\alpha R}\left[I_\mathfrak{z} \ell^2 (I_x + Mh^2) + I_z t^2 (I_x + Mh^2) + M I_x b \ell t^2\right] + V^2\left[M I_x I_z (A \sin \sigma - t C_{\alpha F} \cos \sigma)\right]$$

...

$$D_\Delta = V C_{\alpha F} C_{\alpha R} \cos\sigma \left[ I_z t (I_x + M h^2) + M I_x b l t \right]$$

$$E_\Delta = l^2 C_{\alpha F} C_{\alpha R} \left[ M h g I_{\mathfrak{z}} - A(I_x + M h^2) \sin\sigma \right]$$
$$+ V^2 C_{\alpha F} C_{\alpha R} \left[ M I_x t b \cos\sigma + M h b l \frac{L_F}{R} \cos^2\sigma \right.$$
$$\left. + l (I_x + M h^2)(M_F f + \frac{L_F}{R} \sin\sigma) \right]$$

$$F_\Delta = -V M^2 h g A \sin\sigma (a^2 C_{\alpha F} + b^2 C_{\alpha R}) - V C_{\alpha F} C_{\alpha R}$$
$$\left[ M h t (g I_z - \frac{L_T}{R} V^2) + M h l (g I_{\mathfrak{z}} - \frac{L_F}{R} V^2) \cos^2\sigma \right]$$

$$G_\Delta = - M h g l^2 C_{\alpha F} C_{\alpha R} \sin\sigma \left( A + \frac{V^2 L_F}{l R} \cos\sigma \right)$$

Each of the coefficients in the characteristic expression can be considered individually. Such a review provides an opportunity to examine each of the components of the coefficient for its significance in design and performance.

$$A_\Delta : M I_x I_z I_{\mathfrak{z}} V^2$$

This single element coefficient of the highest order term simply combines the inertial properties of the machine. It is by far the largest of several such terms developed from the complete equations (smaller terms are neglected here). Basically, the designer can work only with the weight and the moment of inertia of the fork assembly ($I_{\mathfrak{z}}$) in influencing performance with this term. Even then, rider weight will have a strong effect on the M, $I_x$ and $I_z$ values. Reduction in $I_{\mathfrak{z}}$ will raise the natural frequency of the castering motion of the assembly but this influence will likely be more sharply felt in other terms of the expression.

$$B_\Delta : V \left[ -M I_x I_z C_{\alpha F} t^2 - M I_x I_{\mathfrak{z}} (a^2 C_{\alpha F} + b^2 C_{\alpha R}) \right.$$
$$\left. - (C_{\alpha F} + C_{\alpha R}) I_z I_{\mathfrak{z}} (I_x + M h^2) \right]$$

These three elements which make up the coefficient of the fifth order term are the damping effects of the steering motion of the fork assembly (the first element) and of the lateral-directional motion of the complete bicycle-rider system (the remaining two elements). They include the inertia coupling influences. The first element, which shows the damping associated with steering rate effects for a positive trailing wheel, could be expanded to include any rate-sensitive damping elements (a viscous steer damper, for example). The negative signs in this term as written are due to the coordinate system chosen for representing tire slip angles; the coefficient is positive when numerical substitutions are made. Here, the designer has control over the amount of trail and tire selection.*

Note that positive trail is required to assure damping of the steering motion. Although several smaller terms have been omitted in this simplified expression (and which could become significant if trail is reduced to near zero) the term demonstrates the importance of positive trail in bicycle design. This point is also shown in the lower order coefficients.

$$C_\Delta \doteq C_{\alpha F} C_{\alpha R} \left[ I_x I_z \ell^2 + I_z M h^2 \ell^2 + I_Z (I_x + M h^2) t^2 + M I_x b \ell t^2 \right] + V^2 \left[ M I_x I_Z (A \sin \sigma - t C_{\alpha F} \cos \sigma) \right]$$

This is the reduced coefficient for the fourth order term. The two major parts consist of the constant (velocity-independent) portion which contains the lateral-directional stiffness effect ($I_z I_x \ell^2 C_{\alpha F} C_{\alpha R}$), a roll moment term ($I_z M h^2 \ell^2 C_{\alpha F} C_{\alpha R}$), and coupling terms, and a velocity-dependent term which is associated with the stiffness of the steering motion dynamics (primarily, $V^2 M I_Z I_x t C_{\alpha F} \cos \sigma$). Thus, we see the fundamental position-control dynamics of the lateral-directional response and the free-control dynamics of the front fork motion appearing in this term. In effect, these primary responses are characterized by the coefficients of the fourth, fifth, and sixth order terms. For the lateral-directional motion, the natural frequency is --

$$\omega_Z = \left( \frac{C_{\alpha F} C_{\alpha R} I_x I_z \ell^2}{V^2 M I_x I_Z I_z} \right)^{1/2} = \left( \frac{\ell^2 C_{\alpha F} C_{\alpha R}}{M I_Z V^2} \right)^{1/2}$$

which is exactly the expression for the fixed control response of a neutral steer vehicle. The steering motion response is described by a natural frequency of --

*Tires are usually selected on bases other than cornering stiffness. Most tires which have been tested at Calspan appear to provide more than adequate cornering capability in dry conditions. See Reference 5.

$$\omega_\delta = \left(\frac{V^2 M I_x I_z (A \sin\sigma - t C_{\alpha F} \cos\sigma)}{V^2 M I_x I_z I_\zeta}\right)^{1/2}$$

$$= \left(\frac{A \sin\sigma - t C_{\alpha F} \cos\sigma}{I_\zeta}\right)^{1/2}$$

$$D_\Delta : V C_{\alpha F} C_{\alpha R} \cos\sigma \left[ I_z t (I_x + M h^2) + M I_x b \ell t \right]$$

The coefficient of the third order term has been reduced from some forty elements to the two shown here. These are coupling expressions which show the influence of trail on damping of the oscillatory motion of the bicycle. The sign of this coefficient is always positive for positive trail values and, for reasonable designs, the values for the two terms are of the same order. The designer has effective influence only over the trail value selection in this coefficient but this value would normally be selected on the basis of other considerations. Again, it should be noted that this coefficient has been greatly simplified from that developed for the original equations of motion and that many small-valued elements (based on reasonable bicycle designs) have been omitted.

$$\begin{aligned} E_\Delta : & - \ell^2 C_{\alpha F} C_{\alpha R} \left[ M h g I_\zeta - A (I_x + M h^2) \sin\sigma \right] \\ & + V^2 C_{\alpha F} C_{\alpha R} \Big[ M I_x t b \cos\sigma + M h b \ell \frac{L_F}{R} \cos^2\sigma \\ & + \ell (I_x + M h^2)(M_F f + \frac{L_F}{R} \sin\sigma) \Big] \end{aligned}$$

This is the reduced coefficient for the second order term and it is one of the most important in determining bicycle stability. Although it still looks quite formidable, the unreduced coefficient contained about six times as many terms. This reduction was again based on omission of small-valued elements as reflected in reasonable designs. The coefficient contains two major parts-- a constant term which is always negative and a velocity-sensitive term which is positive. The coefficient therefore, changes sign (from negative to positive) at some point in the speed range. The operating condition at which the bicycle becomes stable in free-control is closely linked with this point.

All of the design parameters over which the bicycle manufacturer has direct control (trail, head tube angle, wheelbase, fork assembly effects, tire performance) appear in this coefficient. It has not been possible to dissect this coefficient element-by-element in this study but it is noted that the constant term is primarily determined by the value of $A(I_x + Mh^2)\sin\sigma$ for reasonable designs and that the elements in the velocity-sensitive part are of the same order of magnitude. Small values of trail tend to reduce the speed for transition (i.e., change of sign) and small values of front wheel spin inertia tend to increase it. It appears that values for the change-over speed should be in an intermediate range so that low speed instability (associated with high values of the change over speed) is not sustained in the normal operating range and high speed instability (which is the problem for experienced riders) does not occur at too low a value.

$$F_\Delta : \; VM^2hgA\left(a^2C_{\alpha F} + b^2C_{\alpha R}\right)\sin\sigma - VC_{\alpha F}C_{\alpha R}\left[Mht\left(gI_z - \frac{i_T}{R}V^2\right) + Mht\left(gI_\zeta - \frac{i_F}{R}V^2\right)\cos^2\sigma\right]$$

This reduced coefficient of the first order term contains two important design indices. These are the inertia ratios: $I_z/i_T$ and $I_\zeta/i_F$. In addition to the complete coefficient being velocity-sensitive, the effect of these ratios is to cause a change in sign of the coefficient as a function of velocity (going from negative to positive as speed is increased). As with the second order coefficient, most of the easily controlled design parameters are present in this expression.

The effects of the inertia ratio indices are discussed in more detail in a later section. For the purposes here, it may be noted that they can be combined in an expression which can be solved for the speed at which the sign of the coefficient changes (including the third term in the coefficient, which is of opposite sign to that for the $I_z$ and $I_\zeta$ factors and therefore reduces their effect). The influence on stability of this term operates in conjunction with the $S^2$ and $S^0$ coefficients. In the higher speed regime, the time constant of the divergent capsize mode is defined by the ratio of the value of this term to that of the $\overset{c}{S}$ term. In the intermediate range of speed (from 5-15 MPH, in which the sign of this coefficient changes from negative to positive) it represents the damping of this mode.

$$G_{\Delta}: \; -Mhg\,\ell^2 C_{\alpha F} C_{\alpha R} \sin\sigma \left( A + \frac{V^2 \lambda_F}{\ell R} \cos\sigma \right)$$

This term defines the "inversion speed" of the bicycle which was presented and discussed in a more complex form in an earlier study (Reference 2). It relates the gyroscopic moment of the front wheel (stabilizing) to the static upsetting moment of the front fork geometry. At the inversion speed, the two moments are equal for an upright bicycle. The designer has control (within some limits for a given size of machine) over most of the factors in this coefficient.

The value of this coefficient goes from positive to negative with increasing speed. The direction of the applied control steering torque also correlates with this term. It, however, changes sign in reverse order -- at steady-state conditions, applied torque is opposite in sign to steer angle at low speeds and of the same sign at speeds above the inversion speed.

These same terms apply in an analysis of bicycle stability as influenced by rider lean angle. As indicated in Section 2.1, rider lean (with respect to the bicycle frame) can be viewed as simply creating an additional moment (equal to $M_R g k \Phi_R$) in the roll equation. The analysis is based on the use of rider lean angle (rather than on rider lean torque, as used in the model for the simulation) as a convenience for understanding the role of this control function in bicycle dynamics. In this respect, rider lean is treated as an input rather than as a degree of freedom. This simplification allows for the development of easily understandable steady-state transfer functions which are useful for the evaluation of response sensitivities.

The general equations of motion for steering torque control can be portrayed in block diagram form as shown in Figure 3. This figure illustrates

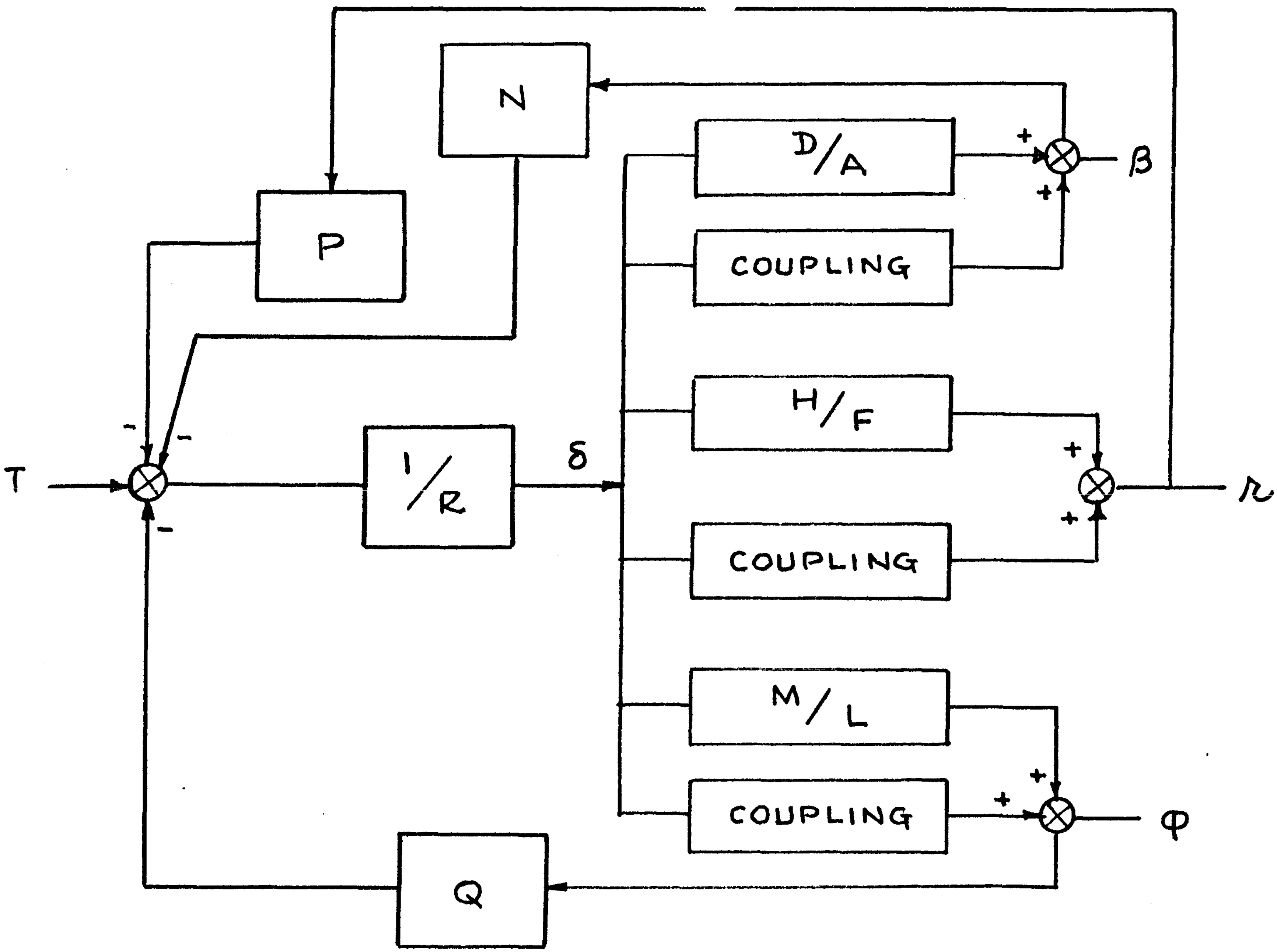


Figure 3: SIMPLIFIED BLOCK DIAGRAM OF BICYCLE SYSTEM

how the simple position control relationships (yaw rate as a function of steer angle, for example) are modified by the motion coupling effects (which are expressed in steer angle terms) and how the resultant motions then influence the steer torque requirements (through the feedback paths). The summing junction at the left of the figure represents the steering torque equation with an output of steer angle as determined by the simple expression for castered wheel motion. The letters in the diagram refer to the designations for the coefficients in the equations of motion as given in Figure 2. The indicated paths from $\delta$ to each of the motion variables consist of one showing the uncoupled relationships and one showing the interactions with the other motions. The transfer functions which represent the coupling terms are rather complicated, containing many terms, and therefore are not completely specified in this simplified diagram.

The discussion in this section of the report has been concerned with the derivation of a constant coefficient expression with which to examine the force-control stability characteristics of the bicycle. Many simplifications have had to be made to reduce the rather formidable (and lengthy) coefficients to a few key terms which permit first order understanding of the dynamics. Most of these simplifications have been made without reservation, and for the most part, the resultant representations are readily interpreted. The coefficient on $S^2$, however, cannot be reduced further without substantial error and we have not been able, up to this point, to define simple relationships which are characteristic of this term. The complete expressions in which all of the interacting terms have been retained are on file at Calspan.

## 2.3 Response Parameters

In addition to a determination of the primary free-control stability of the bicycle, we will be interested in its response to control inputs so that the complete rider-machine system can be analyzed. As part of this study, dynamic transfer functions for the principal motions variables as functions of steering and rider-lean inputs have been determined. However, because of time limitations, it has not been possible to simplify and reduce the lengthy expressions that have been derived (in the same manner that the characteristic expression was simplified). In their present form, they are much too complex to provide useful insight on the specific influences of the various design parameters on performance, and, therefore, they will not be presented here. Instead, only the dynamic expression for steering angle response will be given (to illustrate the general form) and the roll angle and yaw rate responses will be reduced to just the simplified steady-state term. With respect to the identification of control characteristics, the steady-state responses are of primary interest and these will be reviewed in some detail.

Tables 1, 2, 3, and 4 show these transfer functions. Tables 1 and 2 give the complete dynamic equations for steer angle as a function of steering torque and rider lean angle, respectively. Table 3 contains the steady-state roll angle transfer functions with respect to steer angle (position control), steer torque (torque control) and rider lean angle. Table 4 has the same information for the yaw rate responses. These expressions can be used to evaluate the steady-state control gains (sensitivities) of the bicycle.

Examination of the transfer functions for steer angle given in Tables 1 and 2 reveals that the denominators are the same (i.e., the characteristic stability expression does not change) but the numerators are different. In the steady state, the relative effectiveness of applied steering torque and lean angle is defined by the constant,

$$\frac{T}{\Phi_R} = \frac{A M_R h_R}{M h}$$

Table 1: TRANSFER FUNCTION FOR STEER ANGLE TO STEER TORQUE INPUT

$$\frac{\delta}{T} = \frac{A_\delta s^4 + B_\delta s^3 + C_\delta s^2 + D_\delta s + E_\delta}{\Delta_T}$$

$$A_\delta = M I_x I_z V^2$$

$$B_\delta = -V \left[ M I_x \left( a^2 C_{\alpha F} + b^2 C_{\alpha R} \right) + I_z \left( I_x + M h^2 \right)\left( C_{\alpha F} + C_{\alpha R} \right) \right]$$

$$C_\delta = \left( I_x + M h^2 \right) \ell^2 C_{\alpha F} C_{\alpha R} - M^2 h g I_z V^2$$

$$D_\delta = MV \left[ h \left( C_{\alpha F} + C_{\alpha R} \right)\left( g I_z - \frac{i_T V^2}{R} \right) + g \left( a^2 C_{\alpha F} + b^2 C_{\alpha R} \right) \right]$$

$$E_\delta = - M h g \ell^2 C_{\alpha F} C_{\alpha R}$$

$\Delta_T$ = see Section 2.2.

Table 2: TRANSFER FUNCTION FOR STEER ANGLE RESPONSE TO RIDER LEAN ANGLE INPUT

$$\frac{\delta}{\varphi_R} = \frac{A_L s^4 + B_L s^3 + C_L s^2 + D_L s + E_L}{\Delta_L} \quad *$$

$$A_L = V^2 M I_z \left( I_{\zeta} \sin \sigma - M_F f h \right)$$

$$B_L = -V \Big[ M \left(a^2 C_{\alpha F} + b^2 C_{\alpha R}\right)\left(I_{\zeta} \sin \sigma - M_F f h\right) + I_z I_{\zeta} \left(C_{\alpha F} + C_{\alpha R}\right) \sin \sigma + I_z M C_{\alpha F} t h \Big] - M I_z \frac{i_F}{R} \cos \sigma \, V^3$$

$$C_L = \ell^2 C_{\alpha F} C_{\alpha R} M h t + M I_z A V^2 + M C_{\alpha F} \frac{i_T}{R} V^2 + M \frac{i_F}{R} V^2 \cos \sigma \left(a^2 C_{\alpha F} + b^2 C_{\alpha R}\right)$$

$$D_L = -VA\Big[ M \left(a^2 C_{\alpha F} + b^2 C_{\alpha R}\right) + I_z \left(C_{\alpha F} + b C_{\alpha R}\right)\Big] - \ell^2 C_{\alpha F} C_{\alpha R} \frac{i_F}{R} \cos \sigma \, V - V^3 M C_{\alpha F} t \frac{i_T}{R}$$

$$E_L = A \ell^2 C_{\alpha F} C_{\alpha R}$$

$$\Delta_L = \frac{M h}{M_R k} \Delta_T$$

* REDUCED AND SIMPLIFIED

Table 3: STEADY-STATE TRANSFER FUNCTIONS
FOR ROLL ANGLE RESPONSE

$$\frac{\varphi}{T} = \frac{\dfrac{V^2}{\ell g}\cos\sigma \qquad \dfrac{A}{Mhg}}{\sin\sigma\left[A + \dfrac{V^2 i_F \cos\sigma}{\ell R}\right]}$$

$$\frac{\varphi}{\varphi_R} = \frac{A\sin\sigma + V^2\cos\sigma\left[\dfrac{Mbt}{\ell^2} + \dfrac{M_F f}{\ell} + \dfrac{i_F \sin\sigma}{\ell R}\right]}{\sin\sigma\left[A + \dfrac{V^2 i_F \cos\sigma}{\ell R}\right]} \times \frac{M_R k}{Mh}$$

$$\frac{\varphi}{\delta} = \frac{V^2\cos\sigma}{\ell g} + \frac{A}{Mhg}$$

Table 4: STEADY-STATE TRANSFER FUNCTIONS FOR YAW RATE RESPONSE

$$\frac{r}{T} = \frac{\frac{V}{\ell} / \tan \sigma}{A + \frac{V^2 i_F \cos \sigma}{\ell R}}$$

$$\frac{r}{\phi_R} = \frac{\frac{V}{\ell} \cot \sigma}{A + \frac{V^2 i_F \cos \sigma}{\ell R}} \times \frac{A M_R k}{M h}$$

$$\frac{r}{\delta} = \frac{V}{\ell} \cos \sigma$$

It should be noted that the numerator expression given in Table 1 for steer angle response is also the characteristic expression for position control (i.e., it is the denominator for the position control responses of yaw rate and roll angle).

Extracting the steady-state responses from these expressions gives:

$$\frac{\delta}{T} = \frac{1}{\left[A + \frac{\iota_F V^2 \cos\sigma}{\ell R}\right] \sin\sigma}$$

and

$$\frac{\delta}{\varphi_R} = \frac{M_R k A}{M h} \times \frac{1}{\left[A + \frac{\iota_F V^2 \cos\sigma}{\ell R}\right] \sin\sigma}$$

The sign of these expressions changes at the inversion speed, as discussed in Section 2.2. At this speed condition, which is defined by $\left(\frac{A \ell R}{\iota_F \cos\sigma}\right)^{1/2}$, the steer angle response to either applied steering torque or applied lean angle is theoretically infinite. Thus, the further significance of the inversion speed, as it influences controllability of the bicycle, can be appreciated. At operating speeds in the neighborhood of the inversion speed, the control sensitivity is very high (small inputs result in large responses) and this applies not only to the steer angle response but the yaw rate and roll angle responses as well.

The steady-state roll angle response characteristics to T or $\varphi_R$ inputs are marked by two changes in sign over the speed range. As with the other responses, one is determined by the inversion speed. The other arises from the condition for which the numerator term becomes zero. This occurs at a speed defined by --

$$V_\varphi^2 \quad \frac{\ell A}{M h \cos\sigma}$$

for torque control and --

$$V^2_{\varphi R} = \frac{g\, l \tan \sigma}{1 + \dfrac{l\, I_F \sin \sigma}{R\left(M_b t + M_F f l\right)}}$$

for rider lean control.

The steady-state yaw rate response changes as a function of speed in much the same way as the steer angle response does. Sensitivity to both steer torque and rider lean control becomes larger (in a negative sense) up to the inversion speed and then becomes large and positive. The importance of these parameters are associated with bicycle controllability. In the low speed regime, the position control sensitivity ($r/\delta$) is determined by wheelbase and head tube angle. These are also influential in the responses to steer torque and rider lean angle but these latter response sensitivities are also dependent on steering geometry and gyroscopic effects. Values in the low speed regime should be neither too high (excessive control sensitivity) nor too low (producing sluggish behavior). A reasonable range of values for torque control response would appear to be .5 to 1.0 deg/sec/ft-lb. at a low speed (up to 10 MPH), but these values need to be verified by experiment.

To summarize this section it should be pointed out that more study is needed to refine the dynamic transfer functions for yaw rate and roll angle from the standpoint of controllability although the dynamic stability aspect and steady-state gains have been treated. The principal design parameter influences have been identified and the inversion speed index has been shown to be important.

## 2.4 Analysis of Coefficients and Stability Indices

One of the principal objectives of this work was to identify those design characteristics of bicycles which can be used to define their stability and controllability. In general, the stability can be related to the coefficients of the terms in the characteristic expression and an effort has been made to identify factors which dominate these terms. Controllability, to a large extent, is determined by the values of the gain (sensitivity) terms in the steady-state; many of these were developed in Reference 2 and are reviewed briefly here.

The characteristic expression which is developed here contains four combinations of design parameters which are of special interest because of their influence on the response characteristics of the bicycle. These terms are:

Fork Geometry - Gyroscopic Effect

The expression, $A + V^2 \frac{L_F}{R} \cos \sigma$, defines the speed at which the steer torque input requirements theoretically are zero. At this condition, the gyroscopic moment effects just equalize the moment due to steering geometry effects. This speed has been previously identified as the "inversion speed" and it is recognized as a key stability parameter. It appears that acceptable bicycle designs can be categorized in a range of intermediate values. Low inversion speeds (which could result from short trail, light front-end loading, and high wheel spin inertia) are not desirable because they produce divergent instability of the capsize mode at speeds well within the normal operating range. Very high values of this index (if obtained by high front-end loading or high trail values) result in sluggish performance in the normal range.

Inertia Ratios

Two relationships involving moments of inertia are contained in the coefficient of the first-order derivative. They are:

$$g I_z - \frac{L_T V^2}{R}$$

which relates total system moment of inertia about a vertical axis to the moment of inertia of the wheels about their spin axes.

$$g I_{\zeta} - \frac{i_F V^2}{R}$$

which relates the moment of inertia of the fork assembly about the steering axis to the spin axis moment of inertia of the front wheel.

In each expression, the sign of the first term is opposite that of the second term. Each can be solved for a value of speed at which the effect of the parameter is zero (or, at which its value changes sign). In general, this condition occurs at a higher speed for the $I_Z$ parameter than for the $I_{\zeta}$ parameter.

It is more convenient to examine these effects by combining them in a single index. This can be done, using the $F_{\Delta}$ coefficient discussed in Section 2.2, to give a single value of speed at which the combined term changes sign. Noting that $i_T = 2\, i_F$, this speed is--

$$V_G^2 = \frac{\left( I_{\zeta} \ell \cos^2\sigma + I_Z t \right) g R}{i_F \left( \ell \cos^2\sigma + 2t \right)}$$

where the subscript G is used to identify the index as gyroscopic-torque related. This value of speed does not define the condition at which the total first order coefficient changes sign since an additional term involving fork geometry is also present, but (for reasonable designs) this other term is relatively small and can be neglected for purposes of this discussion.

Based on operating experience, it does not appear to be desirable to have a high value for $V_G$. Note that the addition of luggage carriers and steering assembly-mounted equipment tends to raise the values of $I_Z$ or I or both and thereby increases the transition speed, $V_G$. In this respect, the index is related to controllability of the bicycle -- the high inertia values (and high $V_G$) tend to make the bicycle sluggish. From the standpoint of stability, a low value of $V_G$ would appear to be desirable in order to obtain positive values of this coefficient at low operating speeds.

Basically, the bicycle designer can affect this index through the fork geometry factors -- $I_{z}$, $i_F$, $t$, and $\sigma$ for any given size of vehicle (i.e., wheelbase and wheel diameter can be varied only within a small range). It is of interest to compare values of $V_G$ for several Schwinn-designed bicycles. Table 5 below, which is based on data for these units acquired in previous studies, shows these results. To put these values in perspective, $V_G$ has also been computed for two other configurations -- a Suburban with a heavy rear end load and a highly-raked design with large trail (chopper).

Table 5: COMPARISON OF $V_G$ VALUES

| BICYCLE DESCRIPTION | $V_G$ (mph) |
|---|---|
| Schwinn Suburban | 6.7 |
| Schwinn Paramount | 6.3 |
| Schwinn Sprint | 6.0 |
| Schwinn Stingray | 5.8 |
| Chopper (High trail) | 8.2 |
| Suburban (Rear loading) | 7.2 |

## Fork Geometry - Tire Characteristics

The characteristic expression contains a term, $A \sin \sigma - C_{\alpha F}\, t \cos \sigma$, which relates the steering torques due to the geometrical design of the steering assembly to the cornering capability (and self-aligning torque) of the front tire.

The two components of this expression are of opposite sign and their difference defines to first order the spring stiffness of the free control steering response. In conjunction with the moment of inertia of the steering assembly about the steering axis, it defines the natural frequency of the free-control shimmy mode. In effect,

$$\omega^2 = \frac{A \sin \sigma - C_{\alpha F}\, t \cos \sigma}{I_z}$$

Based on a brief review of reasonable bicycle designs, the second element dominates the term and the widely-used simple expression for wheel shimmy of a torsionally-rigid fork assembly can be derived. That is --

$$\omega_n^2 \approx \frac{C_{\alpha F}\, t}{I_z}$$

It is apparent that this term is coupled with other effects (as in the simplified expression of the fourth order coefficient in the characteristic equation) to affect the oscillatory mode dynamic response characteristics of the vehicle. Although it is not quite as clear, this effect also appears in the coefficients of the first and second order terms.

## Bicycle Inertial-Geometric Relationships

As noted previously, the elements in the coefficient of the $S^2$ term do not lend themselves to further simplification, but it is in fact the combination of these elements which plays a major role in defining the character of the oscillatory mode of the bicycle's dynamics. More work is clearly needed on the analysis of these elements since this coefficient, for most designs, determines the range of speed over which the bicycle is stable in free control. This speed range, with regard to its limiting values and its location in the normal spectrum of operation, provides an important performance parameter for bicycle evaluation.

## 2.5 Sample Applications of the Method

The usefulness of the analytical methods developed here depend on how well the simplified model represents actual designs and on its ability to descriminate among different configurations so that potentially unsatisfactory design combinations can be avoided. Although it would be desirable to analyze several different bicycle designs from among many available models, it was believed that the method could be more effectively demonstrated by selecting one model as a reference and varying single design parameters to show differences.

Eight configurational variations of the Schwinn Suburban design parameters which have been used in several previous studies at Calspan were investigated with the constant coefficient model. These included:

(1) Baseline configuration. Values for the primary parameters of this configuration are given in Table 6. A 160 pound rider was used for all cases.

(2) Short trail configurations. Keeping all other values fixed, the mechanical trail was reduced from 3 inches to 1 inch.

(3) Long trail configuration -- mechanical trail increased to 4 inches.

(4) Steep head tube angle configuration. With all other parameters as in the baseline configuration, the head tube angle was increased from 69 degrees to 74 degrees. ( Steer axis caster angle changed from 21 to 16 degrees.)

(5) Shallow head tube angle configuration -- reduced angle to 64 degrees. (Steer axis caster angle changed to 26 degrees.)

(6) Low wheel spin inertia configuration -- Substitution of wheels with spin inertia values of .73 in-lb-sec$^2$ for the baseline

## TABLE 6

BASELINE CONFIGURATION FOR 22 INCH SCHWINN SUBURBAN BICYCLE

| | |
|---|---|
| WHEELBASE (IN) | 41.50 |
| CASTER ANGLE OF THE STEER AXIS (DEG) | 21.00 |
| NOMINAL STEERING TRAIL (IN) | 3.00 |
| PERPENDICULAR DISTANCE FROM C.G. OF FRONT FORK ASSEMBLY TO STEER AXIS (IN) | 1.50 |
| HEIGHT OF TOTAL C.G. ABOVE GROUND (IN) | 39.54 |
| LOCATION OF TOTAL C.G. FORWARD OF THE REAR WHEEL CENTER (IN) | 16.57 |
| TIRE ROLLING RADIUS (IN) | 13.60 |
| FRONT TIRE CORNERING STIFFNESS (LB/DEG) | -14.19 |
| REAR TIRE CORNERING STIFFNESS (LB/DEG) | -18.46 |
| | |
| TOTAL WEIGHT OF BICYCLE AND RIDER (LB) | 200.80 |
| WEIGHT OF FRONT FORK ASSEMBLY (LB) | 11.40 |
| TOTAL ROLL MOMENT OF INERTIA ABOUT AN AXIS THROUGH THE TOTAL C.G. (LB-IN-SEC SQ) | 134.22 |
| TOTAL YAW MOMENT OF INERTIA ABOUT AN AXIS THROUGH THE TOTAL C.G. (LB-IN-SEC SQ) | 42.39 |
| YAW MOMENT OF INERTIA OF FRONT FORK ASSEMBLY ABOUT THE STEER AXIS (LB-IN-SEC SQ) | 1.86 |
| SPIN MOMENT OF INERTIA OF THE FRONT WHEEL (LB-IN-SEC SQ) | 1.76 |
| SPIN MOMENT OF INERTIA OF THE REAR WHEEL (LB-IN-SEC SQ) | 1.76 |

values of 1.76 in-lb-sec$^2$.

(7) Low tire pressure configuration -- Replacement of baseline tire cornering performance characteristics with values corresponding to Puff tires at 20 psi inflation pressure.*

(8) Low steering assembly moment of inertia configuration -- Yaw moment of inertia of front fork assembly about the steer axis reduced from 1.86 in-lb-sec$^2$ to .71 in-lb-sec$^2$.

This array provides reasonable coverage of the limiting design conditions recommended for ISO adoption on head tube angle and trail and includes an evaluation of wheel and tire influences. The reduced wheel spin moments of inertia are those for the Paramount bicycle (to provide a frame of reference). The low tire pressure configuration was included to evaluate possible effects from reduced tire cornering capability. Results are given in Table 7.

The performance characteristics given in Table 7 can be briefly explained in the framework of bicycle stability and control as discussed below:

1. The "speed range for unconditional stability" characterizes the bicycle's best operating range. It should be reasonably broad (certainly spanning a range of several miles per hour) and situated in the band of normal operation. The baseline configuration meets these criteria very well but the short trail design (No. 2) would place a heavy burden on the rider to provide stabilization except in a small region of operation. Design No. 6, the low spin moment of inertia configuration (characteristic of better handling bikes), places this band higher in the speed range. The lower limit of this band appears to depend on the inertia parameters ( $I_z$ and $I_{\mathfrak{z}}$), the upper limit on the gyroscopic effects.

*Performance data based on Calspan tire tests reported in Reference 5.
$C_{\alpha F}$ = -9.44 lbs/deg.; $C_{\alpha R}$ = -7.38 lbs/deg.

Table 7: SAMPLE COMPUTATION RESULTS

| | CONFIGURATION | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| Speed range for unconditional stability (MPH) | 10.5-17 | 9-10.5 | 11.5-20 | 10-16.5 | 11-17.5 | 14-26 | 11-17 | 10-17 |
| Speed for onset of oscillatory instability (MPH) | 36 | 14.5 | 46 | 37 | 35 | 47 | 27 | 57 |
| Performance at 12 MPH | | | | | | | | |
| • Oscillatory mode frequency (Hz) | .59 | 1.24 | .46 | .65 | .54 | .37 | .54 | .58 |
| • Oscillatory mode damping (% of critical) | .34 | .03 | .12 | .42 | .22 | -.16 | .38 | .44 |
| Capsize mode time constant at 20 MPH (secs) | 9 | 3.5 | 38 | 11 | 7.5 | --- | 9 | 9 |
| Frequency of oscillatory mode at zero damping condition | 3.8 | 1.6 | 4.3 | 3.7 | 3.8 | 4.1 | 2.5 | 5.4 |

2. The parameter "speed for onset of oscillatory instability" gives a partial measure of the high speed stability of the bicycle. It is of course, desirable that its value be above the maximum normal operating speed. Since this oscillation occurs at frequencies above rider control capabilities for all configurations considered (see line 3), designs for which oscillation occurs at operating speeds are to be avoided. Again, the short trail configuration (No. 2) is poor.

3. The 12 MPH operating speed condition has been selected for evaluating stability in normal operation. All configurations appear to be rideable at this condition but the configurations 2 and 6 require considerable rider compensation. The short trail design (No. 2) is approaching free-control instability and the low steering assembly moment of inertia design (No. 6) has not yet reached the speed for free-control stability but the oscillation frequency is low enough that rider compensation can be utilized to produce stable system response.

4. The capsize mode time constant (evaluated at 20 MPH) gives an indication of behavior in the divergent mode of instability. High values for this parameter (of several seconds) are desirable to allow the rider adequate time for compensation. In fact, experienced riders are probably not conscious of providing this compensation on reasonably-designed bicycles. In any case, this effect does not appear to present any problems in the normal range of speed through which most of these configurations are unconditionally stable with the exception of the short trail design (No. 2).

5. The values for the "frequency of the oscillatory mode at zero damping" are given to demonstrate that, if operating conditions are reached for which this mode becomes unstable (generally, at high speed as indicated by the second line in the table), the frequency of the oscillation is above that for which the rider can

apply effective control. Even though the indicated frequencies for Configurations 2 and 7 are relatively low with respect to the others, they are believed to be beyond the bandwidth for which human controllers are normally capable. Thus, it is important that bicycles be designed to place this condition well beyond the normal range of speed.

The numerical computations in this study were limited to those for a 160 lb. rider in the upright riding position. Changes in rider weight and position (particularly as they affect values for $I_Z$, $I_X$, M, h, and $Z_F$) will result in changes in the values of the stability indices given here. The magnitude of these changes, with respect to the reference values and the differences obtained by design variations, can be used to demonstrate the sensitivity of specific designs to these operational factors.

Another aspect which should be considered in greater depth is the relative importance of position control in bicycle riding. To place this point in context, recognize that automobiles are primarily controlled in normal operation by steering position rather than applied steering torque. The driver supplies whatever torque is required to achieve the desired steer angle displacement for a particular maneuver. Only under extreme conditions (emergency maneuvering, steering system failure) does torque control play an important role. In two-wheel vehicles, steering control displacements to cover the complete range of operation are small and the rider depends on torque control rather than position control.

In an attempt to put some of these results into perspective with regard to their application to regulatory standards, Figure 4, which completely illustrates proposed design limitations now under consideration by the International Standards Organization (ISO), has been prepared. The eight bicycle configurations which were analyzed earlier in this section are located on the diagram by number. Note that several of the configurations occupy the same location as defined by the two simple constraints used even though their theoretical performance characteristics (as shown in Table 7) are quite dif-

NOTE: Diagram based on Reference 7.

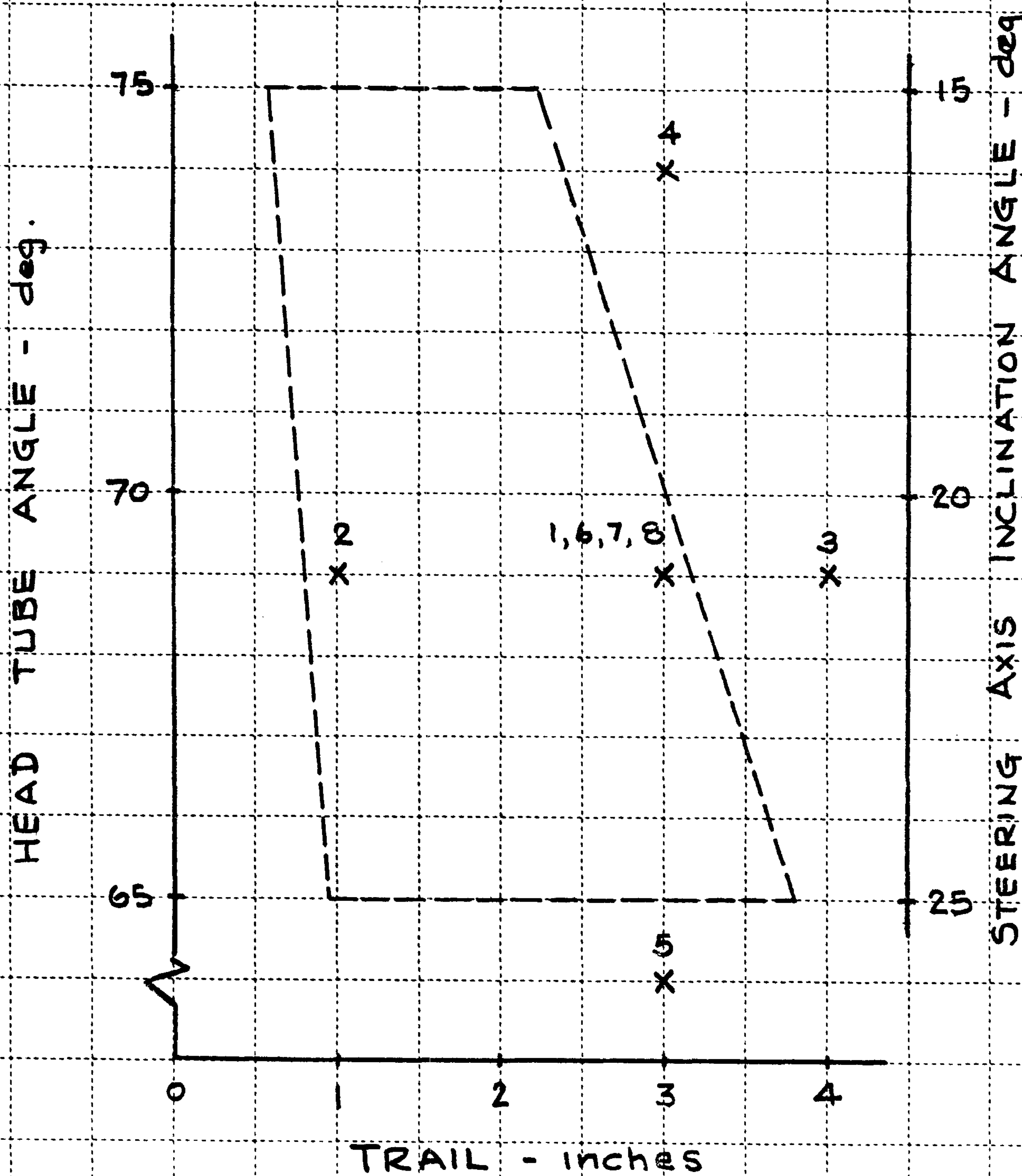


Figure 4: DESIGN CONFIGURATION COMPARISON BASED ON PROPOSED ISO STANDARD

ferent. The following observations can be made --

1. Configurations 3, 4, and 5 would be unacceptable according to the ISO criteria. Yet, the performance index approach, according to the values given in Table 7, indicate that No. 5 is very much like the baseline configuration (No. 1) and that Nos. 3 and 4 are not greatly different from stability considerations.

2. Configuration No. 2 (the low trail design) is within the acceptability bounds on the diagram but appears to present substantial control problems at high normal speeds according to the performance analysis.

3. Since the proposed ISO approach does not address tire characteristics and moment of inertia effects, potential performance differences which are implied by the stability analysis (and which surely exist on actual machines) are not adequately treated. In effect, the proposed technique does not discriminate against poor combinations of these other design parameters.

## 3. CONCLUSIONS

Steady-state stability analyses performed in an earlier study have been extended to cover the complete dynamic response of the bicycle in this report. Constant coefficient linear equations of motion in four degrees of freedom have been examined in an effort to identify key design factors with which to characterize bicycle stability. Four primary interacting design parameters have been isolated. They are:

- the ratio of fork geometry and weight distribution effects to the front wheel moment of inertia about the spin axis.
- the ratios of the moments of inertia of the fork assembly and rider-bicycle system about the steering and yawing axes to the wheel spin moments of inertia.
- the relationship between wheel loads and tire cornering forces.
- the inertial parameters and size factors which influence the primary oscillatory motion mode.

In turn, these parameters have led to the definition of several stability indices which characterize bicycle performance in terms of specific speed values. These indices are:

- the inversion speed (the speed at which the capsize mode goes unstable)
- the minimum free control speed (the speed at which the steer and roll responses are compatible)
- the oscillatory mode critical speed (the speed at which the damping of the oscillatory mode goes through zero)

- the absolute stability speed range (over which the free-control bicycle is stable)

Sets of steady-state response parameters relating motion output to control input have also been developed. In conjunction with the stability indices, these response sensitivity terms provide a means for bicycle evaluation on a performance basis.

In addition, it has been shown how a number of individual bicycle design factors affect stability according to this performance analysis technique. Among the principal conclusions regarding these are:

1. Small values of positive mechanical trail and all negative values of mechanical trail are to be avoided in bicycle design if satisfactory free control stability is to be achieved. Specific limits cannot be defined without the support of experimental test data, but an observation that proposed ISO standards permit designs with trail values which are too small for use by unskilled riders appears justified. Tentatively, a lower limit in the range of 1.5 to 2.0 inches for 27 inch wheel bicycles is suggested.

2. In the normal operational speed range, head tube angle (in the range of 75 to 60 degrees) has little effect on free control stability characteristics (other parameters being held constant). This is not to say that limits should not be defined (very steep angles will reduce the value of the sin $\sigma$ term and will limit trail values without substantial fork design changes, for example) but the acceptable range appears to be larger than the 65-75 degrees indicated in the suggested ISO standard.

3. The use of low spin moment-of-inertia wheels is not good practice on bicycles to be used by novice riders. This parameter affects the speed range over which the bicycle is hands-off stable; low values raise the lower limit (and increase the range) so as to

improve elevated speed stability but place a heavier burden on the rider for stabilization at low speeds.

4. Tire cornering performance characteristics (at least when used in matched sets) affect stability primarily in the higher speed range. Inferior cornering capability in the tires showed up as reducing the speed at which the oscillatory mode becomes unstable and, in general, providing less damping of this mode at all conditions in the high normal speed range.

Thus, with respect to the early evaluation of the performance of projected new designs, the analytical method described herein provides some cues for selecting suitable combinations of the design parameters. In application to the problem of formulating performance standards, the method has enabled the selection of several factors -- response parameters and stability indices -- which may define requirements with sufficient rigor that all unacceptable configurations can be identified.

But it is apparent that we do not yet know where to place the limits on these factors to assure good overall performance of the machine. Specific answers are needed to the following questions --

1. What are the correlations between values of the stability indices and subjective ratings of the bicycle's performance over the operating envelope?

2. What "rules of thumb" can be formulated about stability and controllability tradeoffs?

3. What are practical test methods for evaluating these performance parameters?

## 4. RECOMMENDATIONS

This report outlines an analysis of a constant coefficient model of bicycle dynamics. Several combinations of design parameters, called stability indices, have been identified and an attempt to show how they influence stability and control has been made. It is recommended that consideration now be given to determining ranges of values for these indices which are representative of satisfactory performance. Such a study would involve full-scale testing of several bicycle configurations (including some having characteristics which would be suspect based on the analyses given here) and using several riders of different skill levels. Both objective and subjective evaluation methods should be employed. An efficient test program, producing results which would not only be applicable to the questions raised here but also to the further validation of the bicycle dynamics simulation program, can be readily devised. In fact, it would be based on an outline of suggested experimental work previously submitted to Schwinn (Reference 8, Task 4). The utilization of 2 or 3 basic frame configurations with readily interchangeable fork assemblies of varying physical characteristics (to enable independent evaluations of the several design parameters of interest) would be advantageous. This, too, has been previously suggested (Reference 8, Task 3).

At this time, the approach to be taken would consist of the following --

- List the design and operating parameters of importance, and the ranges of each to be covered, based on the analytical results given here.
- Devise test methods and identify instrumentation requirements which would be effective in the determination of the stability indices and response parameters defined in this report.
- Perform pilot tests with a reference bicycle to refine test techniques.

- Perform test program with several bicycle configurations using the refined testing techniques.

- Based on the test results, recommend ranges of acceptable values for the performance characteristics (stability indices and response parameters).

- Concurrently, apply the test data in conjunction with the simulation of bicycle dynamics to improve understanding of rider lean control.

In summary, recommended definitions of acceptability limits on bicycle performance (after the manner of the proposed ISO standard or, better, given in performance terms rather than design terms) should be obtained by supplementing the analytical results of this study with experimental data which could be transformed into representative values of the response parameters and stability indices that have been discussed.

# 5. REFERENCES

1. Roland, R. D. & Rice, R. S., Bicycle Dynamics -- Rider Guidance Modeling and Disturbance Response. Calspan Report No. ZS-5157-K-1, April 1973.

2. Rice, R. S., Bicycle Dynamics -- Simplified Steady-State Response Characteristics and Stability Indices. Calspan Report No. ZN-5431-V-1. June 1974.

3. Sharp, R. S., The Stability and Control of Motorcycles. Journal of Mechanical Engineering Science, Vol. 13, No. 5. 1971.

4. Fu, Hiroyasu, Fundamental Characteristics of Single-Track Vehicles in Steady Turning. Bulletin of JSME, Vol. 9, No. 34. 1965.

5. Davis, J. A., Bicycle Tire Testing -- Effects of Inflation Pressure and Low Coefficient Surfaces. Calspan Report No. ZN-5431-V-3. May 1975.

6. Kunkel, D. T. & Rice, R. S., Low Speed Wobble Study of the Harley Davidson Electra Glide FLH-1200 Motorcycle. Calspan Report No. ZN-5473-V-1. July 1975.

7. Anon., Testing Methods and Performance Requirements for Cycles and Their Assemblies. ISO/TC 149 SCI (UK-1). June 1973.

8. Anon., Bicycle Dynamics -- Suggestions for Continuing Research. Calspan Corporation, no number. July 1975.

# APPENDIX A
## TABLE OF SYMBOLS

The symbols for the various mathematical quantities used in this report are defined in this section. Many of them were shown in Figure 1 of the body of the report. For the most part, they require no special explanation; however, it should be kept in mind that the sign convention used in the analyses results in negative cornering stiffness values (i.e., $C_{\alpha F}$ and $C_{\alpha R}$ are negative quantities, in keeping with Society of Automotive Engineers practice) and that the steering axis inclination (head tube) angle is measured from the vertical.

| | | | |
|---|---|---|---|
| $I_X$ | - | moment of inertia of rider-bicycle system about horizontal longitudinal axis through c.g. | $\text{lb-ft-sec}^2$ |
| $I_Z$ | - | moment of inertia of rider-bicycle system about vertical axis through c.g. | $\text{lb-ft-sec}^2$ |
| $I_{\mathfrak{z}}$ | - | moment of inertia of steering assembly about the steer axis. | $\text{lb-ft-sec}^2$ |
| M | - | total system (bicycle plus rider) mass. | $\frac{\text{lb-sec}^2}{\text{ft}}$ |
| $M_R$ | - | rider upper body mass. | $\text{lb-sec}^2/\text{ft.}$ |
| T | - | applied steering torque. | ft-lb. |
| V | - | forward velocity. | ft/sec |
| $F_{yF}$, $F_{yR}$ | - | tire side force (front and rear) | lbs. |
| $M_F$ | - | steering assembly mass | $\frac{\text{lb-sec}^2}{\text{ft}}$ |
| $Z_F$ | - | ground reaction force at front wheel contact patch | lbs. |
| $C_{\alpha F}$, $C_{\alpha R}$ | - | tire stiffness coefficients (front and rear) | lbs/rad. |
| R | - | tire rolling radius. | ft. |
| a | - | horizontal distance between center of front wheel and total system c.g. | ft. |
| b | - | horizontal distance between center of rear wheel and total system c.g. | ft. |

| | | | |
|---|---|---|---|
| f | - | front fork mass offset; perpendicular distance from steer axis to steering assembly, c.g. | ft. |
| g | - | gravitational constant | ft/sec$^2$ |
| h | - | vertical distance from road surface to total system c.g. | ft. |
| i | - | wheel moment of inertia about its spin axis ($i_F$ - front wheel; $i_T$ - both wheels) | lb-ft-sec$^2$ |
| k | - | vertical distance between rider upper body c.g. and the upper body pivot point. | ft. |
| l | - | bicycle wheelbase. | ft. |
| r | - | yaw rate. | rad/sec. |
| t | - | mechanical steering trail, perpendicular distance from steer axis to center of front wheel contact patch (positive, as defined) | ft. |
| $\alpha$ | - | tire slip angle. | rad. |
| $\phi$ | - | bicycle roll angle. | rad. |
| $\beta$ | - | bicycle sideslip angle. | rad. |
| $\delta$ | - | steering assembly displacement angle. | rad. |
| $\sigma$ | - | steering axis inclination angle. | rad. |
| $\phi_R$ | - | rider lean angle. | rad. |

NOTE: Signs are based on a right-handed coordinate system.

# APPENDIX B
## SAMPLE APPLICATIONS

This appendix contains two examples of how the roots of the characteristic expression can be plotted to provide a comparison of the stability properties for different bicycle configurations. Its purpose is merely to demonstrate how the method lends itself to a detailed mathematical analysis of performance should such investigations be desired. It provides a somewhat more complete picture of stability and control over the operating speed range of the bicycle than the tabulated results given in the main body of the report but it is also more difficult to interpret without being familiar with this manner of presentation.

Figure B-1 shows root locus plots for two bicycle configurations -- (1) the Schwinn Suburban as manufactured, and (2) a similar model with a much reduced value of trail of one inch. The better performance of the Suburban is clearly shown by the more extensive loop described by the oscillatory mode locus in the left side of the diagram. The short trail design simply does not develop sufficient damping in this region of operation. To facilitate interpretation, the loci show arrows which indicate how the roots change with increasing speed and the values for the oscillatory mode at speeds of 10, 20, and 30 MPH are identified by the encircled symbols. Two real roots are also depicted -- one which increases with increasing speed with a nominal time constant in the range of .15 to .07 seconds and the other which decreases with increasing speed and moves into the right half plane at the inversion speed.

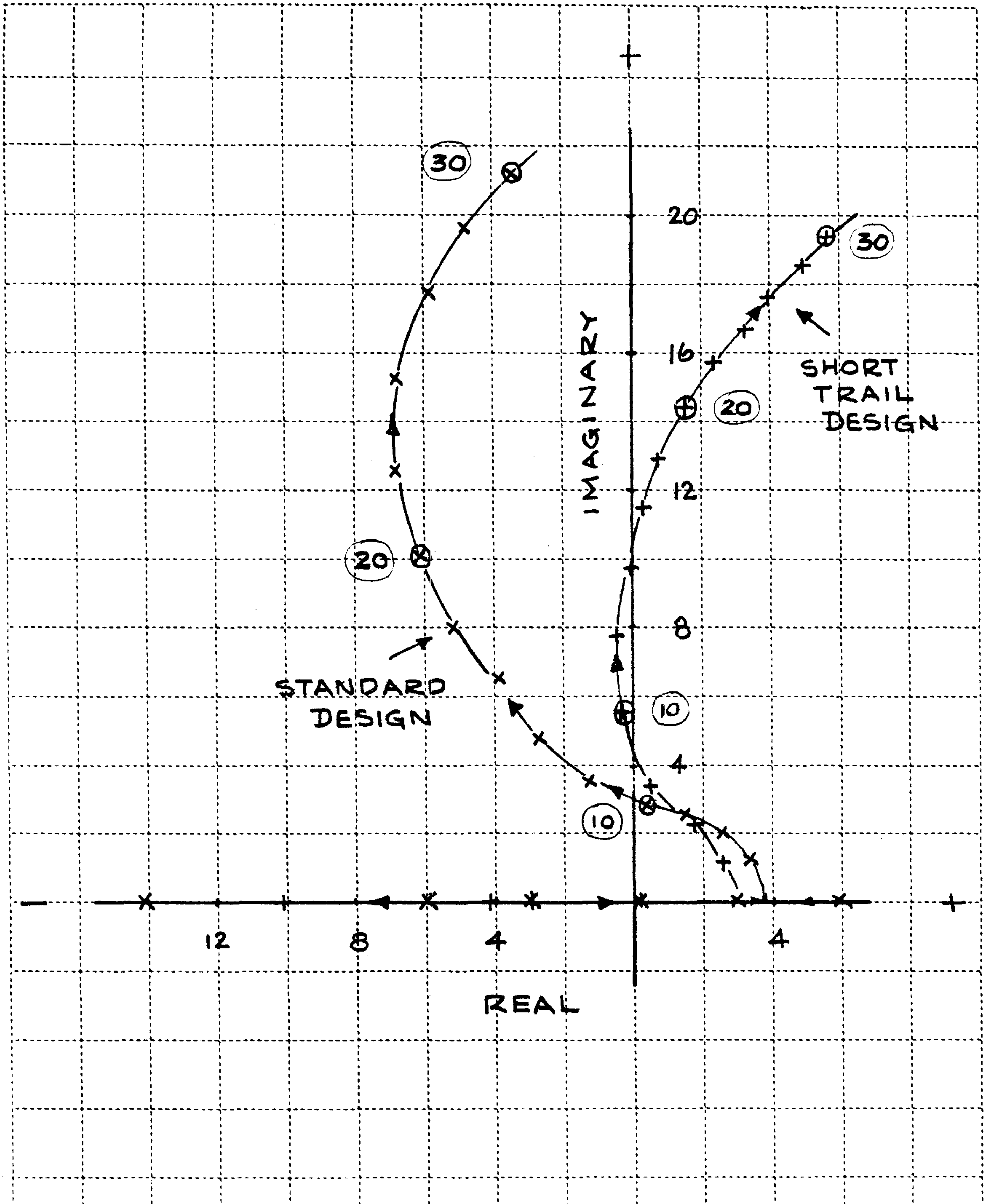


Figure B-1: EXAMPLES OF ROOT LOCUS PLOTS