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Design Sensitivity Analysis of Bicycle Stability and Experimental Validation

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ABSTRACT

Based on the linear form of uncontrolled bicycle dynamic equations, sensitivity of weave and capsize speed with respect to design parameters are calculated. Analyzing sensitivity curves, significance of design parameters along with the way how stable speed range is changed is found. Among 7 significant parameters out of 25, head angle is the most dominant parameter followed by front wheel diameter, mass, moment of inertia demonstrating the importance of front side design in bicycle stability. Procedure for predicting stable speed range using sensitivity information is investigated. When a single parameter is changed, the stable range is determined by that parameter, meanwhile when multiple parameters are changed, stable range is determined by adding up all contributions from each parameter. Using an experimental bicycle with variable configuration, the weave speeds at nominal design and changed configuration that gives lowest value of weave speed are measured. The comparison between the measured and predicted value of weave speed shows good correlation, which demonstrates the validity of the sensitivity based stability analysis.

Keywords: Bicycle Dynamics, Self Balancing, Sensitivity Analysis

1 INTRODUCTION

Balancing of bicycles has been the subject of numerous scientific studies by many researchers since the inception of bicycles [1]. Bicycle balancing can be categorized into two groups according to nature of balancing; balancing by controlling rider and uncontrolled balancing by bicycle itself. A rider can balance a forward-moving bicycle by turning the front wheel in the direction of lean, which consequently moves the tire contact points with ground toward the direction of lean similar to balancing of an inverted pendulum. Moreover, the centrifugal force due to circular motion caused by steering also contributes to balancing. On the other hand, uncontrolled bicycles can balance themselves within some velocity range [2], which is dependent on various design parameters of bicycles..

To understand driver behaviour by controlling rider to balance a bicycle, Lee et. al. [3] measured the steering and leaning of rider with respect to a bicycle in recovering balance when unexpected sudden lateral force is applied to the straight moving bicycle. It was found that the rider's steering and leaning motion is dependent on the lateral acceleration and roll angle, and control law for balancing was derived based on this observation.

Takehara et. al. [4] studied the effects of tire size and offset on the stability of portable, foldable, and compact bicycles with small tires mainly for city use using 3-dimensional multibody model. They found that larger tire is better for stability, however that tendency is strongly dependent on the forward velocity. This result is quite consistent with the fact that the uncontrolled bicycle is stable for some velocity interval.

Meijaard et. al. [2] reviewed extensive list of bicycle dynamic formulations since the start of bicycle dynamics study around 200 years ago, and checked their correctness. They proposed a 2-

dof linear bicycle dynamic model with 25 parameters, and demonstrated stable velocity range for self balancing through eigen value analysis. Based on the formulation by Meijaard et. al., Kooijman [5] performed experimental validation on the stable range of velocity for self balancing.

The scope of this work is to investigate the effect of design parameters of bicycles on the uncontrolled self balancing. For this, the linear form of equations of motion by Meijaard [2] along with the definition of 25 design parameters is adopted. Sensitivity of stable range with respect to design parameters is computed and the significance of design parameters along with their tendency in design change is investigated. Sensitivity based design procedure to alter range of stability is proposed. Also through an experiment with a variable configuration experimental bicycle, proposed design procedure for lowering lower limit of stability, which is weave velocity, is validated.

2 Bicycle Dynamics

The bicycle model has four bodies; a rear wheel, a rear frame with the rider body rigidly attached, a front frame including handle bar and fork assembly, and a front wheel, as shown in Figure (1). The bodies are connected by revolute joints at the steering head between the rear frame and the front frame and at the two wheel axes. In the reference configuration, all bodies are assumed to be symmetric relative to the bicycle mid-plane. In this model wheels are circular disk without thickness.

A bicycle moves on flat surface and wheels make point contact with the road without slip. The contact between the wheels and the surface is modelled as stiff and non-slipping holonomic constraints in the normal direction and by non-holonomic constraints in the longitudinal and lateral direction. It is assumed that there is no friction, apart from the idealized friction between the non-slipping wheels and the surface, and no propulsion. These assumptions make the model energy conservative.

The bicycle has two degrees of freedom, rear frame roll angle ϕ, and steering angel δ, thus the generalized coordinate vector **q** can be expressed as

$$\mathbf{q} = [\phi, \ \ \delta]^{\mathrm{T}} \tag{1}$$

The constant forward velocity v is given as

$$v = -\dot{\theta} R_{rw} \tag{2}$$

where R_{rw} is the radius of the rear wheel and $\dot{\theta}$ is angular velocity. The total of 25 design parameters for the experimental bicycle are defined in Table 1, and they are represented by a design parameter vector **b** as

$$\mathbf{b} = [b_1, b_2, \ldots, b_{25}]^T \tag{3}$$

The linearized equations of motion at the vicinity of ϕ=0, δ=0 can be expressed as [2]

$$\mathbf{M}\ddot{\mathbf{q}} + v\mathbf{C}_1\dot{\mathbf{q}} + [g\mathbf{K}_0 + v^2\mathbf{K}_2]\mathbf{q} = \mathbf{f} \tag{4}$$

where **M** is a symmetric mass matrix, $v\mathbf{C}_1$ a damping matrix linear in the forward speed v, $g\mathbf{K}_0$ is a stiffness matrix which is the sum of a constant and symmetric part proportional to gravity g, and $v^2\mathbf{K}_2$ is part of stiffness matrix quadratic in the forward speed, and **f** represent applied torque vector associated with roll angle and steer angle.

$$\mathbf{f} = \begin{bmatrix} T_\phi & T_\delta \end{bmatrix}^{\mathrm{T}} \tag{5}$$

Transient response of bicycle without any external force can be expressed by a linear combination of eigen vectors. For this, the solution of Equation (1) is assumed to be an exponential form as

$$\mathbf{q} = \mathbf{q_0} \exp(\lambda t) \tag{6}$$

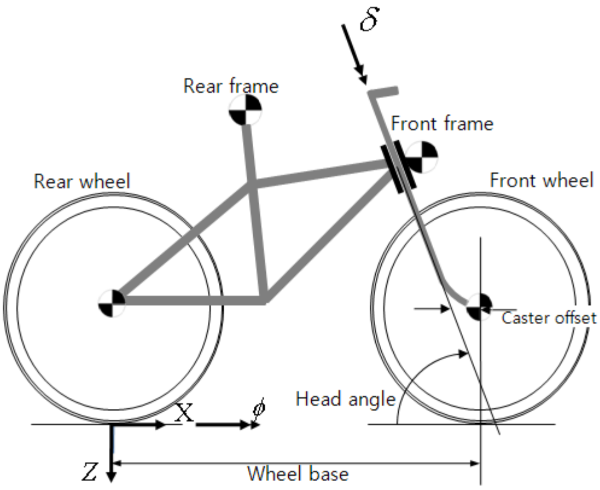


Figure 1. Geometry of bicycle and design parameters

Table 1. Definition of design parameters for the experimental bicycle

No.	Symbol	Definition	Value [unit]
1	w	Wheelbase	1.06 [m]
2	α	Head angle	70.0 [16deg]
3	ε	Caster offset	0.02 [m]
4	D_{fw}	Diameter of front wheel	0.678 [m]
5	D_{rw}	Diameter of rear wheel	0.678 [m]
6	m_{fw}	Mass of front wheel	2.18 [kg]
7	m_{rw}	Mass of rear wheel	2.77 [kg]
8	m_{ff}	Mass of front frame	3.97 [kg]
9	m_{rf}	Mass of rear frame	22.27 [kg]
10	d_{rf}	Distance rear frame mass center -rear wheel	0.3 [m]
11	h_{rf}	Height of rear frame mass center	0.5 [m]
12	d_{ff}	Distance front frame mass center-rear wheel	0.883442 [m]
13	h_{ff}	Height of front frame mass center	0.64835 [m]
14, 15, 16	A_{xx}, A_{yy}, A_{zz}	Mass moments of inertia of rear wheel	(0.07721, 0.16432, 0.07721) [kgm^2]
17, 18, 19	B_{xx}, B_{yy}, B_{zz}	Mass moments of inertia of rear frame	(0.855397, 1.817005, 1.637713) [kgm^2]
20, 21,22	C_{xx}, C_{yy}, C_{zz}	Mass moments of inertia of front frame	(0.25818, 0.22212, 0.051123) [kgm^2]
23, 24,25	D_{xx}, D_{yy}, D_{zz}	Mass moments of inertia of front wheel	(0.07925, 0.15123, 0.07925) [kgm^2]

Characteristic polynomial equation can be obtained as

$$\det\left(\mathbf{M}s^2 + v\mathbf{C}_1 s + g\mathbf{K}_0 + v^2\mathbf{K}_2\right) = 0 \tag{7}$$

Eigen values with positive real part correspond to unstable motions whereas eigenvalues with a negative real part correspond to asymptotically stable motions for the corresponding mode. Imaginary eigenvalues correspond to oscillatory motions.

There are four eigen modes, where oscillatory eigen modes come in pairs. Among them two modes, capsize mode and weave mode are significant. The capsize mode corresponds to a real eigenvalue with eigen vector dominated by lean: when unstable, the bicycle just falls over like a capsizing ship. The weave mode is an oscillatory motion in which the bicycle sways about the headed direction. The third remaining eigen mode is the caster mode which corresponds to a large negative real eigenvalue with eigenvector dominated by steering.

Figure 2 shows values of eigen values for the experimental bicycle. The speed at which the eigen value for weave mode becomes negative is defined as weave speed v_w, and speed at which the eigen value for capsize mode becomes negative is defined as capsize speed v_c. Thus the stable speed range for uncontrolled bicycle is between v_w and v_c.

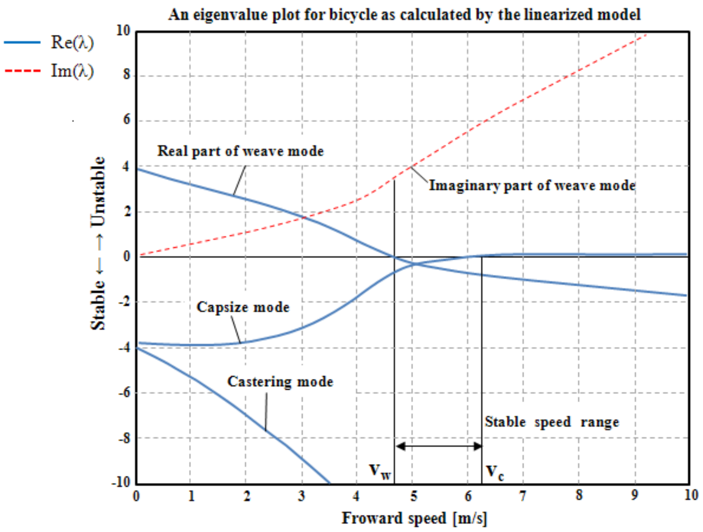


Figure 2. Eigenvalues λ from the linearized stability analysis of the experimental bicycle.

3 Sensitivity analyses for stability

The weave speed v_w and capsize speed v_c, which depend on the design parameters **b,** determines the stable range of an uncontrolled bicycle. Thus if the sensitivity of weave and capsize speed with respect to each design parameter is known, stable range can be altered.

Assume that a design parameter b_i is changed by δb_i, then its new value b_i' is given as

$$b_i' = b_i + \delta b_i , \tag{8}$$

With the changed design parameter, new value of weave and capsize speed, v'_w and v'_c, can be given as

$$v_w' = v_w'\left(b_1, \cdots, b_{i-1}, b_i + \delta b_i, b_{i+1}, \cdots, b_n\right) \tag{9}$$

which can be computed by solving Equations (1) and (7). Figure 3 shows 25 sensitivity curves for weave and capsize speed with respect to 25 individual design parameters defined in Table 1. y-axis represents design parameter change within ±20% range. For each design parameter, there is a pair of sensitivity curves; the left side curve represents weave speed v_w and the right side curve represents capsize speed v_c for the corresponding design parameter.

Sensitivity curves in Figure 3 can be categorized into three groups, Type I, II, and III, according to their shape or equivalently effectiveness on altering stable range. Figure 4 shows typical shapes for Type I, II, and III sensitivity curves, and in Table 2, the design parameters are calssified into 3 groups.

For Type I sensitivity curves, when the parameters are changed, both weave speed v_w and capsize speed v_c are decreased or increased simultaneously. Larger gradient of sensitivity curve means bigger change in v_w and v_c. In case of Type II sensitivity curves, v_w and v_c are almost independent of parameter changes, thus effect of design parameters in this group is insignificant. For Type III sensitivity curves, v_w remains almost constant while v_c is decreased or increased.

Head angle(α), caster offset(ε), wheel base(w), mass of front wheel(m_{fw}), diameter of front wheel(D_{fw}), front wheel moment of inertia with respect to hub(D_{yy}) belong to Type I, and distance from front frame mass centre to rear wheel (d_{ff}) is the only Type III parameter, and the others are Type II. Thus, from the sensitivity curves, it can be observed that among 25 parameters, 18 parameters that belong to Type II are negligible and 6 parameters for Type I and 1 parameter for Type III are significant meaningful design parameters. Moreover all Type I and III parameters are directly related to the front frame and wheel, which demonstrates the importance of front side regarding bicycle stability.

As shown in Type I and III sensitivity curves in Figure 4, head angle (α) is the most dominant design parameter, and second level significant parameters are front wheel moment of inertia with respect to hub(D_{yy}), mass of front wheel(m_{fw}) and diameter of front wheel(D_{fw}), where these three sensitivity curves are almost the same since these parameters associated with front wheel are strongly coupled to each other. Since the most dominant parameter is head angle, it can be observed that stable region could be shifted down with narrower range of stability or stable region could be shifted up with wider range of stability.

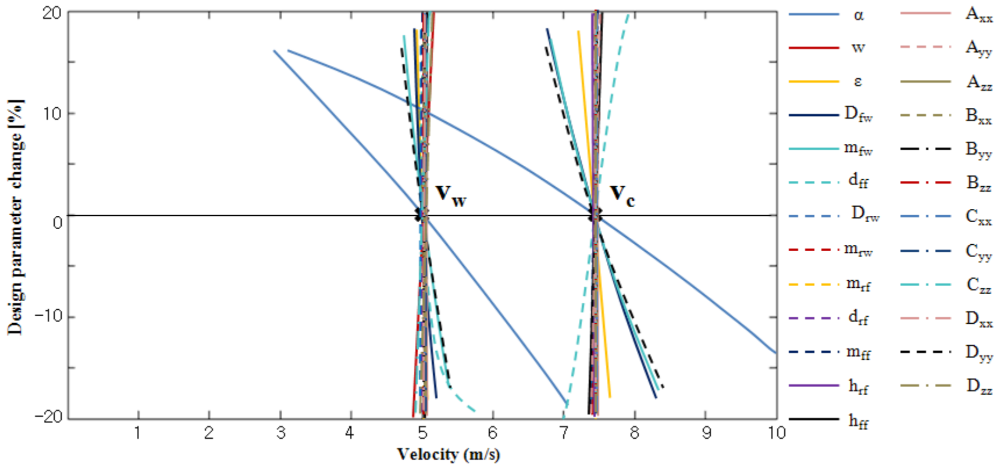


Figure 3. Sensitivity curves for weave and capsize speed on the 25 design variable at the nominal design configuration of the experimental bicycle.

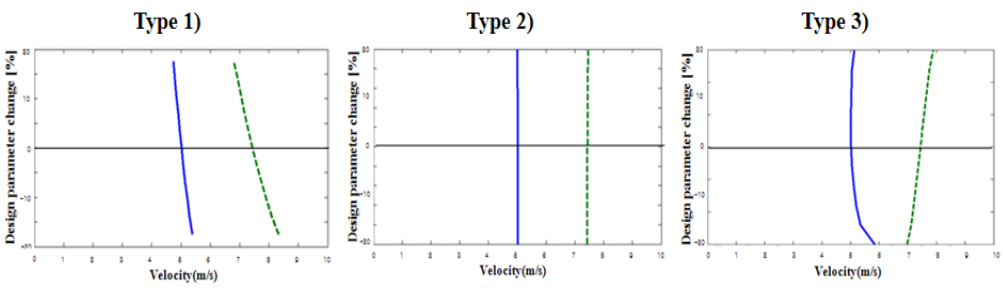


Figure 4. Three types of sensitivity curves

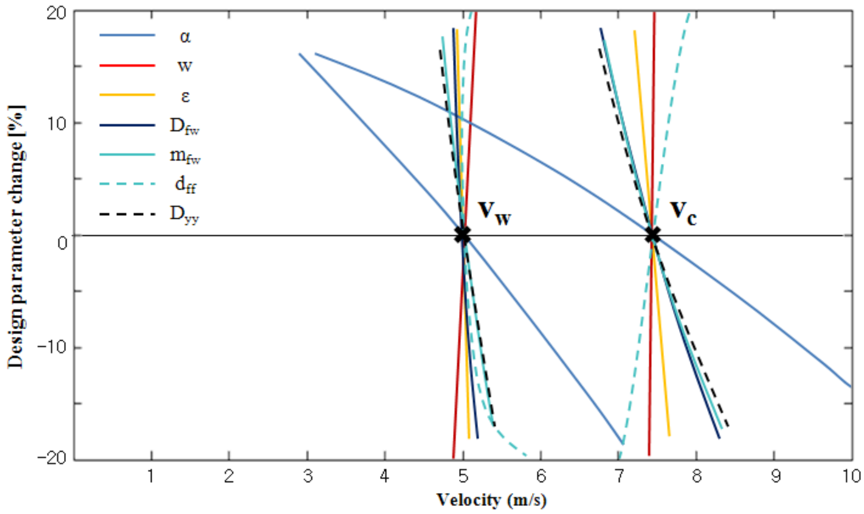


Figure 5. Sensitivity curves for Type I and Type III

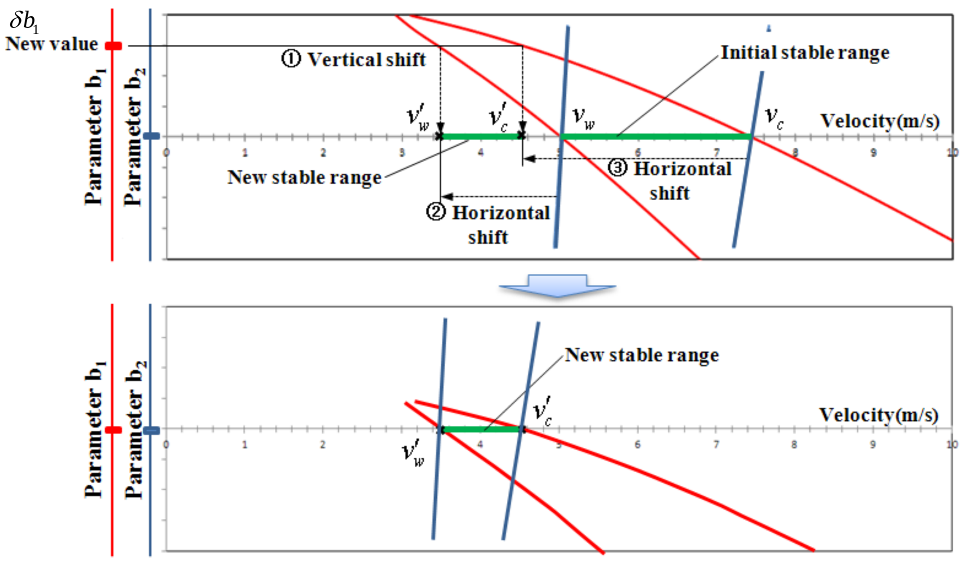


Figure 6. Procedure for predicting weave and capsize speed and sensitivity curves with single parameter change.

Table 2 Classification of design parameter according to sensitive curve shape

Type	Design parameters
Type I	α, ε, w, D_{fw}, m_{fw}, D_{yy}
Type II	A_{xx}, A_{yy}, A_{zz}, B_{xx}, B_{yy}, B_{zz}, C_{xx}, C_{yy}, C_{zz}, D_{xx}, D_{zz}, h_{rf}, d_{rf}, m_{ff}, m_{rf}, D_{rw}, m_{rw}, h_{ff}
Type III	d_{ff}

4 Prediction of stable range

Utilizing the sensitivity information, stable range between weave speed v_w and capsize speed v_c can be predicted with design parameters change. Single parameter or multiple parameters can be changed. First of all, let's consider a case where one parameter, for example, head angle, the most dominant parameter, is changed. If the head angle is increased to a new value of

$b_i' = b_i + \delta b_i$ as shown in Figure 6, then the stable range is changed from the initial range(v_w ~ v_c) to new range (v'_w ~ v'_c). Since all the sensitivity curves should pass through the two points v_w and v_c regardless of design parameter values, other sensitivity curves are shifted to the newly determined stable range.

The procedure for determining new stable range and sensitivity curves is to be explained in detail. First design variable b_1 is changed to $b_1' = b_1 + \delta b_1$, and corresponding new stable range in the pair of sensitivity curve for weave and capsize speed for b_i is determined. Second, the pair of sensitivity curve for b_1 is down shifted to the x-axis since design is changed to new value $b_1' = b_1 + \delta b_1$. Third, sensitivity curves for weave speed and capsize speed for other pairs of sensitivity curves are respectively shifted horizontally to new v'_w and v'_c .

It should be pointed out that at the new design configuration, $\left(b_1, \cdots b_{i-1}, b_i + \delta b_i, b_{i+1} \cdots, b_n\right)$, sensitivity curves may differ from those of at the original configuration. Thus this analysis is valid with linear shape sensitivity curves and small range of parameter change. In general sensitivity curves for any type of bicycle are quite similar to Figure 3 and linear.

The procedure for single parameter change can be extended to the multiple parameter changes. Since all the design parameters are independent, changed v'_w and v'_c should reflect contributions from each design parameter. Thus by summing up all the contributions from individual design parameter change, v'_w and v'_c can be computed as

$$v_w' = v_w + \sum \left(\Delta v_w\right)_{b_i}, \qquad v_c' = v_c + \sum \left(\Delta v_c\right)_{b_i} \qquad (10)$$

Figure 7 demonstrate change of stable range when three design variables b_1, b_2 and b_3 are changed. Due to the change of b_1, weave speed and capsize speed is respectively reduced by 0.48 and 0.85 [m/s], for b_2, weave speed and capsize speed is respectively increased by 0.07 and 0.18 [m/s], and for b_3, weave speed and capsize speed is respectively reduced by 0.01 and 0.02 [m/s]. If all changes are summed up, weave speed is decreased by 0.51 and capsize speed is decreased by 0.69. Then each pair of sensitivity curve is fitted into the new stable range first by vertical shift of weave and capsize sensitivity curve, followed by respective individual horizontal shift of weave and capsize sensitivity curve to v'_w and v'_c.

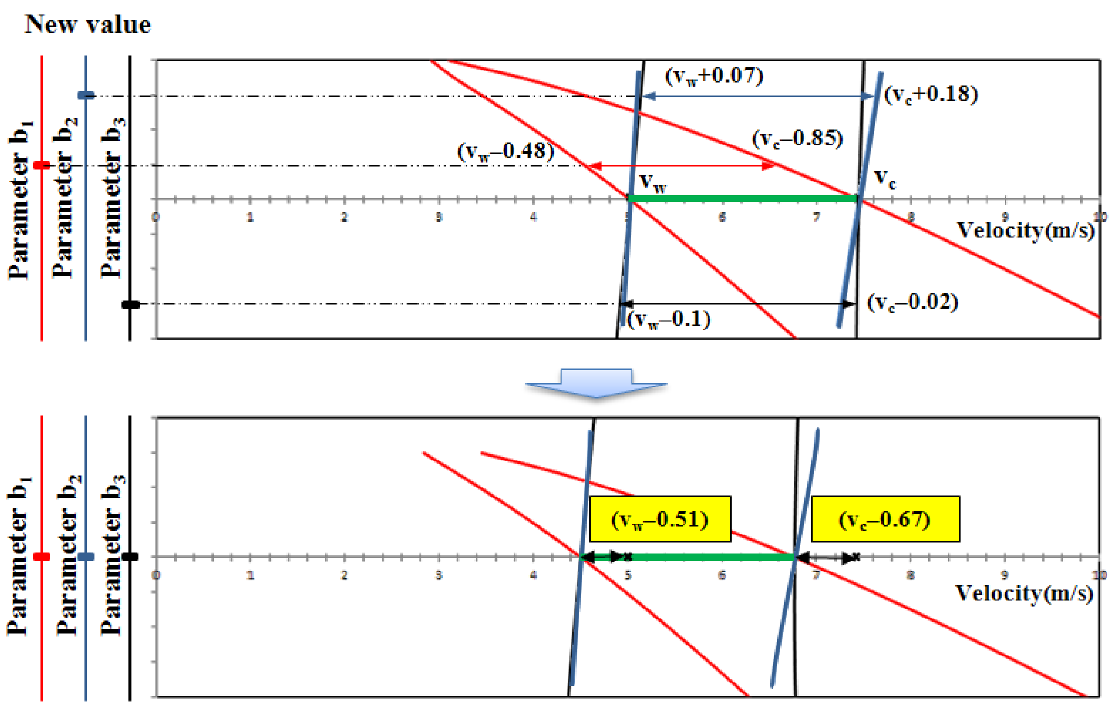


Figure 7. Stable range change and shift of sensitivity curves due to multiple parameter change.

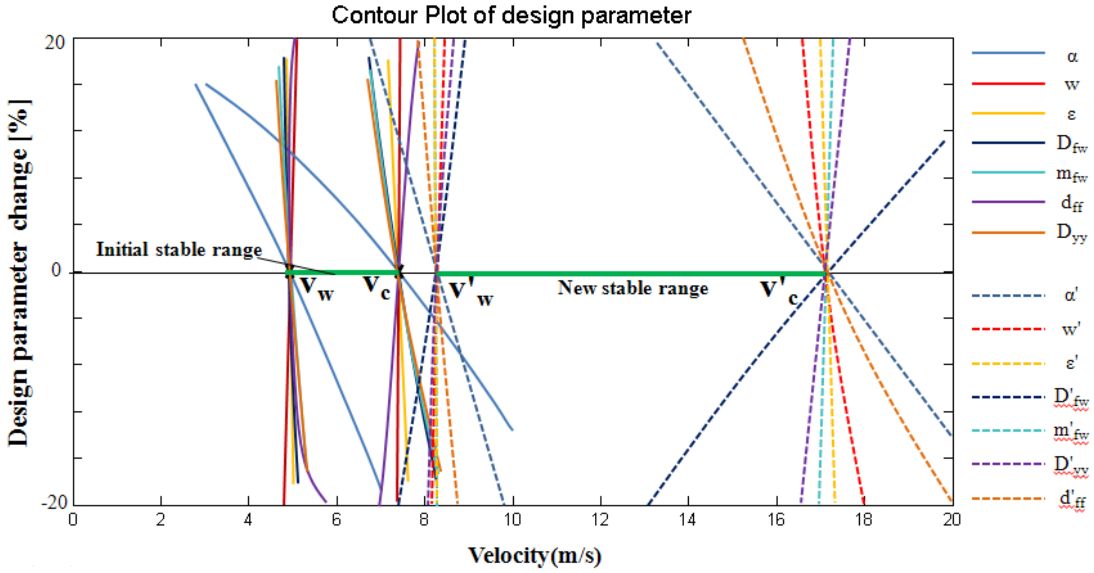


Figure 8. Extended stable range and sensitivity curves

Table 3. Design parameter values for maximum stable speed range

#	Symbol	Sensitivity Type	Definition	Initial Value [unit]	Final Value [unit]
1	w	Type I	Wheelbase	1.016 [m]	0.8160 [m]
2	α		Head angle	70.7822 [deg]	56.7822 [deg]
3	ε		Caster offset	0.0272 [m]	0.0218 [m]
4	D_{fw}		Diameter of front wheel	0.6858 [m]	0.8230 [m]
5	m_{fw}		Mass of front wheel	1.1818 [kg]	2.1818 [kg]
6	D_{yy}		Mass moment of inertia of front wheel	0.1710 [kgm^2]	0.1368 [kgm^2]
8	d_{ff}	Type III	Distance front frame mass center –rear wheel	0.8834 [m]	1.0601 [m]

Excluding Type II parameters, using 7 Type I and III parameters, maximum stable range is calculated within ±20% of parameter change. The initial stable range of 5.1581~7.9509 [m/s] is extended to 8.3330-16.9131 [m/s]. Figure 8 shows initial and new stable ranges along with corresponding sensitivity curves, and at Table 3 design parameter values for maximum and initial stable speed range are compared.

5 Experimental Validation

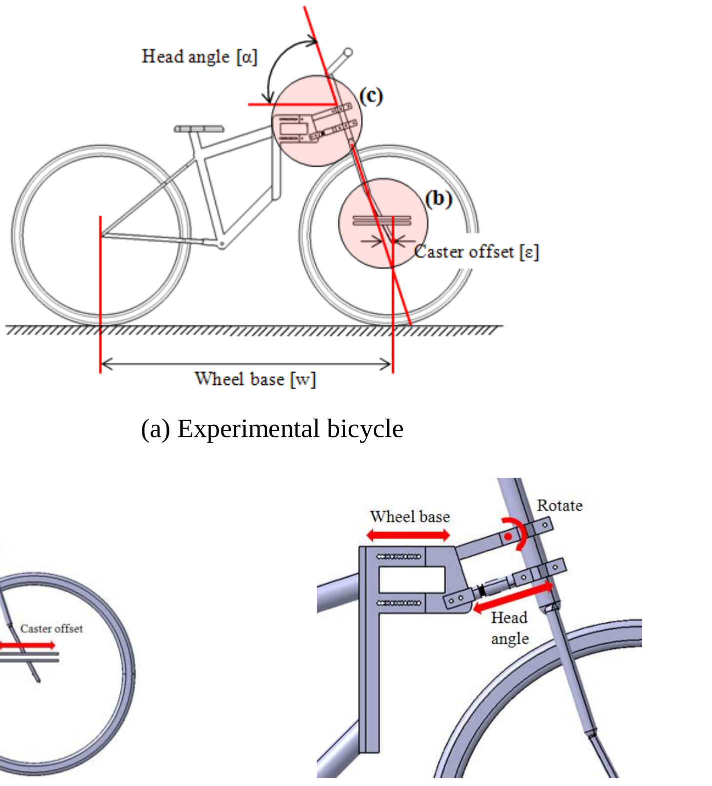
To validate stable range changes depending on parameter, an experimental bicycle that can independently adjust wheel base, caster offset, head angle are designed and manufactured as shown in Figure 9. Head angle can be changed by altering the length of lower link, which is part of a 4 bar mechanism comprising steering head and rear frame, and wheel base can be lengthened or shortened by periscope type parallel mechanism, and caster offset can be adjusted by a sliding mechanism attached to the fork. The range of change is ±6°for head angle, between +0.06m and −0.04m for wheel base and ±0.04m for caster offset.

Steering wheel angle, lateral acceleration, wheel rotational speed, body lean angle against bicycle and rear frame roll angle are measured with the experimental bicycle shown in Figure 10. A rotary sensor is attached at the handle to measure steering angel, and lateral acceleration is measured by an accelerometer at the bicycle CG point, a beam sensor is adopted for measuring wheel rotational speed and a tilt sensor measures rear frame roll angle. To measuring body lean angle, a long slender stick was fitted to the axis of the rotary sensor, and the end of stick was attached to the torso belt. Data acquisition PC and battery are loaded at the rear rack.

According to the stable range design procedure, the lowest value of weave velocity is obtained by changing head angle, caster offset and wheel base within the allowable limit of the experimental bicycle. The weave speed at the original configuration is 5.49 m/s, and with change of head angle by +6°, wheel base by -0.04m, and caster offset by +0.01m, respectively, the weave speed is reduced to 4.46 m/s.

The weave speed is measured for both initial and changed configuration of the experimental bicycle. Test is carried out on a flat surface shown in Figure 11. On a straight lane of width of 0.5m, a rider is asked to accelerate bicycle to reach above 7m/s and maintain the speed before crossing the line. After crossing the line the rider stops pedalling and take off both hands from the handle and stay motionless suppressing any active balancing movement using any part of body. If the rider feels bicycle becomes unstable, he grabs the handle and stops the bicycle.

During the test, roll angle, steer angle and velocity of bicycle are measured. The moment when bicycle gets unstable is judged from the roll angle and steer angle. Figure 12 and 13 shows two sets of steer and roll angles to judge moment of instability for initial and changed configurations. For each configuration, 10 tests are tried and after discarding results which are not clear or difficult to determine the moment of instability, average values of weave velocity at the time of instability are obtained. The results are summarized in Table 4, and it can be observed that the error between the calculated and measured weave speeds for nominal and changed configuration is small.

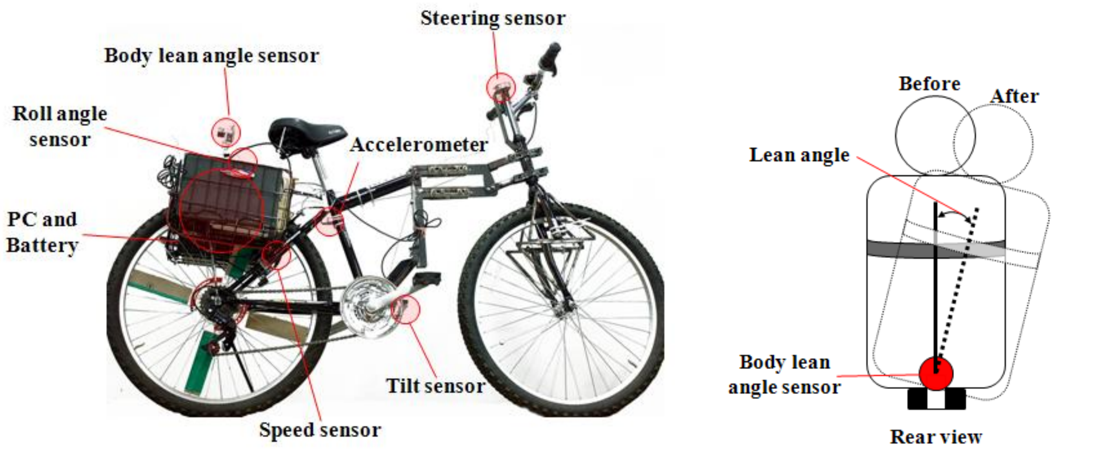


(a) Experimental bicycle

(b) caster offset control part

(c) Wheel base and head angle control part

Figure 9. Experimental bicycle with variable configuration



(a) data acqusition system for experimenntal bicycle (b) lean angle measurement

Figure 10. Data acquisition system of the experimental bicycle.

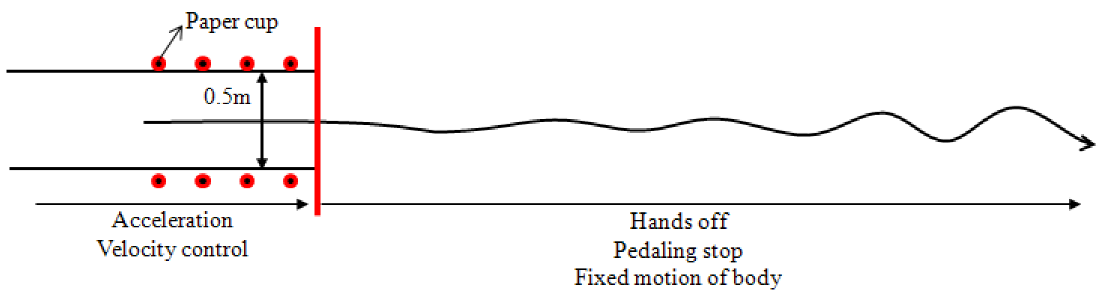
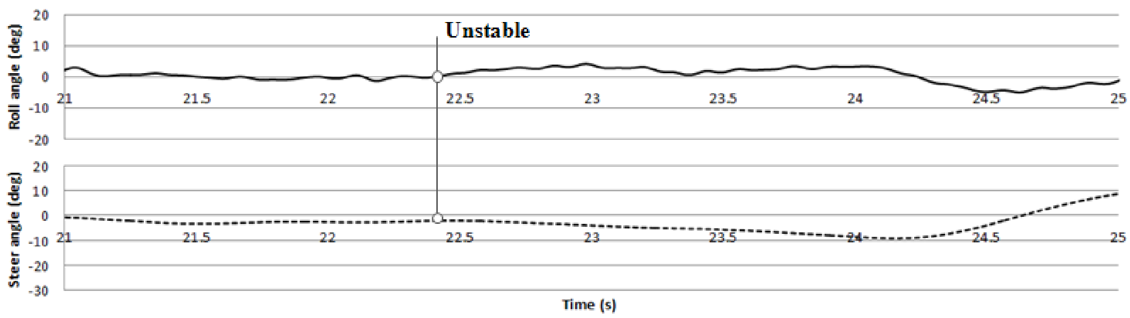
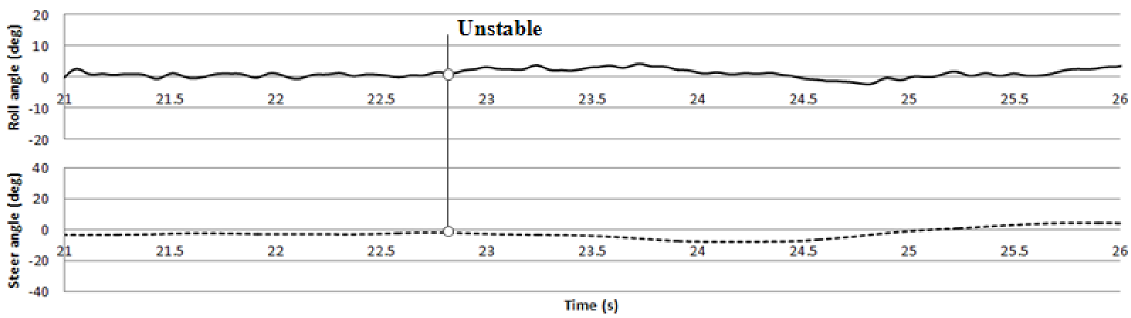


Figure 11. Course of the uncontrolled bicycle test

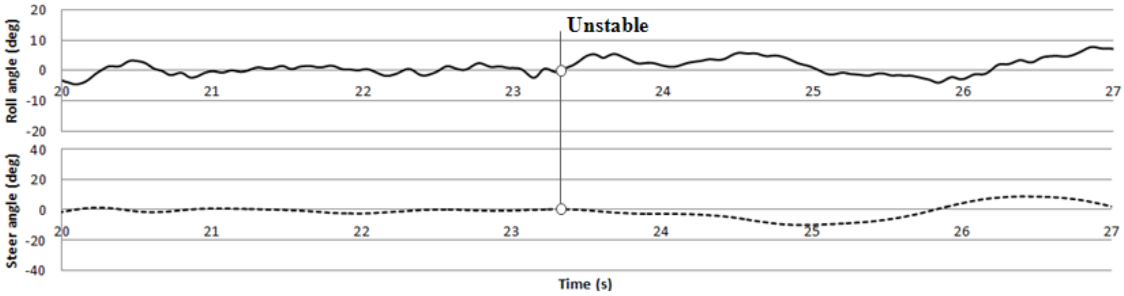


(a) Example of determining unstability in terms of roll and steer angle

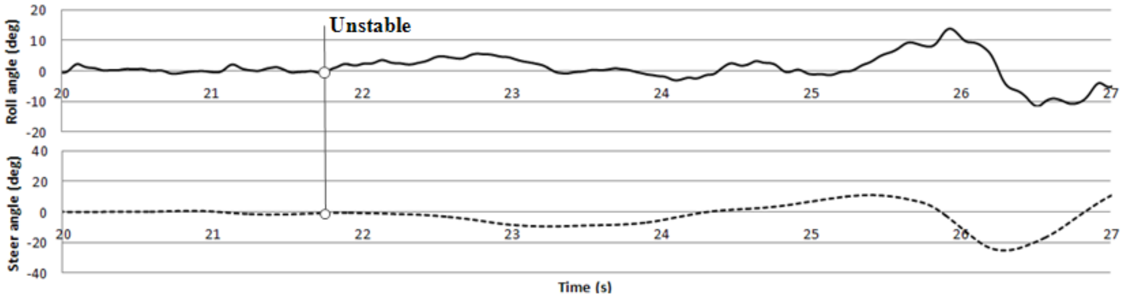


(b) Example of determining unstability in terms of roll and steer angle

Figure 12. Determination of instability for nominal configuration bicycle with weave speed of 5.49 m/s



(a) Example of determining instability in terms of roll and steer angle



(b) Example of determining unstability in terms of roll and steer angle

Figure 13. Determination of instability for changed configuration bicycle with weave speed of 4.46 m/s

Table 4. Comparison of the measured and calculated weave speed

configuration / method	Calculated values [m/s]	Experimental values [m/s]
Nominal parameters	5.49	5.37
Changed parameters	4.46	4.66

7 CONCLUSIONS

Based on the linear form of uncontrolled bicycle dynamic equations, sensitivity of weave and capsize speed with respect to design parameters are calculated. Sensitivity curves are classified according to their shape or effectiveness on stable range change. Among 25 design parameters, 18 parameters are negligible and 7 parameters are significant; head angle is the most dominant followed by front wheel moment of inertia with respect to hub, mass of front wheel and diameter of front wheel, which are all directly related to the front part of bicycle. Due to the shape of sensitivity curves of the angle, stable region could be shifted down with narrower range of stability or stable region could be shifted up with wider range of stability

Procedure for predicting stable speed range using sensitivity information is investigated. When a single parameter is changed, the stable range is determined by that parameter, meanwhile when multiple parameters are changed, stable range is determined by adding up all contributions from each parameter. It is demonstrated that by changing design parameters within allowable limit, stable range can be considerably expanded.

An experimental bicycle that can alter head angle, wheel base and cater offset is designed and manufactured. The lowest weave velocity is obtained by changing head angle, caster offset and wheel base within design change limit. Using the experimental bicycle, the weave speeds at nominal configuration and changed configuration that gives lowest weave speed has measured. The comparison between the measured and predicted value of weave speed shows good correlation, which demonstrates the validity of the sensitivity based stability analysis.

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