Proceedings, Bicycle and Motorcycle Dynamics 2010
Symposium on the Dynamics and Control of Single Track Vehicles,
20 - 22 October 2010, Delft, The Netherlands

A bicycle model for education in machine dynamics and real-time interactive simulation

J.L. Escalona, A.M. Recuero

Department of Mechanical and Materials Engineering
Camino de los Descubrimientos, s/n, 41092, Sevilla, Spain
e-mail: escalona@us.es, amrecuero@us.es

ABSTRACT

This paper describes the use of a bicycle model to teach machine dynamics. The bicycle equations of motion are first obtained as a DAE system written in terms of dependent coordinates that are subject to holonomic and non-holonomic constraints. The equations are obtained using symbolic computation. The DAE system is transformed to ODE system written in terms of a minimum set of independent coordinates using the generalized coordinates partitioning method. This step is taken using numeric computation. The ODE system in then linearized about the upright position and eigenvalue analysis of the resulting system is performed. The frequencies and modes of the bicycle are obtained as a function of the forward velocity which is used as continuation parameter. The resulting frequencies and modes are compared with experimental results. Finally, the non-linear equations of the bicycle are used to create an interactive real-time simulator using Matlab-Simulik. Some issues about controlling the bicycle are discussed. All the paper is focused on teaching engineering students the practical application of analytical and computational mechanics using a model that being simple is familiar and attractive to them.

Keywords: education of machine dynamics, symbolic computation, real-time simulator.

1 INTRODUCTION

A bicycle *Whipple model* is an excellent example to teach machine dynamics to engineering students. The model can be used from intermediate level to high level courses. This model allows students to better understand analytical dynamics of constrained mechanical systems as well as computational techniques with a system that is familiar to them. The bicycle model while being simple contains a large variety of ingredients that make it very attractive for teaching purposes. This model let students to enjoy a subject (Machine Dynamics) that may result difficult to understand and separated from reality in a first approach. In what follows it is explained the mechanical properties of the bicycle model under an analytical point of view and the ideas that students can learn with it.

The bicycle model is described with a set of 9 coordinates that must fulfil all kind of kinematic constraint equations:

- Holonomic and non-holonomic constraints.
- Scleronomic and rehonomic constraints.
- Joint, contact and driving constraints.

The definition of contact constrain equations requires the use of a geometric parameter to define the position of the wheel contact point with the ground. This parameter is a non-generalized coordinate of the system since no inertia is associated with it. Therefore the bicycle model includes:

- o Dependent generalized coordinates.
- o Non-generalized coordinates.

The equations of motion of the system are obtained in this work using Lagrange equations. Due to the existence of non-holonomic constraints and the use of dependent coordinates, *Lagrange equations of the first kind* that include *Lagrange multipliers* to account for reaction forces are needed. Due to this fact the resulting equations of motion are *differential-algebraic equations* (DAE). Although the dimension of the problem is not large (9 coordinates) pencil and paper calculations of the equations of motion is prohibitive. In this work the DAE equations of motion are obtained using computer symbolic calculation.

Although multibody dynamic techniques allow the use of the DAE equations of motion for linearization and eigenvalue analysis [1] it is more appropriate to transform first the DAE system to ordinary differential equations (ODE) with which the students are more familiar. This transformation is done in this paper by the *generalized coordinate decomposition method* [2] that transforms the equations to a new set written in terms of independent coordinates and eliminates the Lagrange multipliers from the system. This transformation of the system equations from ODE to DAE can no longer be done symbolically because of the nature of the algebraic manipulations involved. Therefore the transformation of the equations is done numerically and the rest of analyses that follow are also performed numerically. Students can understand also the difference between numerical and symbolic computer calculations.

Once the equations are written in ODE form in terms of independent coordinates, the equations are linearized around the equilibrium position, being such a position in this case the upright position although circular trajectories can also be selected. After linearization an eigenvalue analysis of the system is performed using the bicycle forward velocity as continuation parameter. The regions of stability are identified. The stable region of the continuation plot is experimentally verified using the method described in [3].

The bicycle non-linear DAE equations of motion can be integrated numerically forward in time. The symbolic computations performed are well suited for real-time simulation allowing developing an interactive simulator. The simulator uses peripherals such joysticks used in computer games to virtually riding the bicycle. However, the input signal that can be used to control the bicycle is not trivial. Different possibilities are discussed although the control input remains an open problem.

2 KINEMATICS

This section explains the coordinates selected to describe the bicycle and the constraints that they must fulfil due to kinematic joints, wheel-ground contact and forward velocity.

2.1 Coordinates Selection

Figure 1 shows a drawing of the bicycle in an arbitrary position. The bicycle is made of 4 rigid moving bodies. Solid 2 is the rear wheel, solid 3 is the frame, solid 4 is the handlebar and solid 5 is the front wheel.

Frame <*X Y Z*> is the global-inertial frame of reference to which the position and orientation of all moving bodies are refered. Frames <x_{i1} y_{i1} z_{i1}> and <x_{i2} y_{i2} z_{i2}> are intermediate frames needed to define the plane that contains the bicycle frame. Finally, each moving body has its own body frame <x_i y_i z_i >, i = 2, 3, 4 and 5, as shown in the figure.

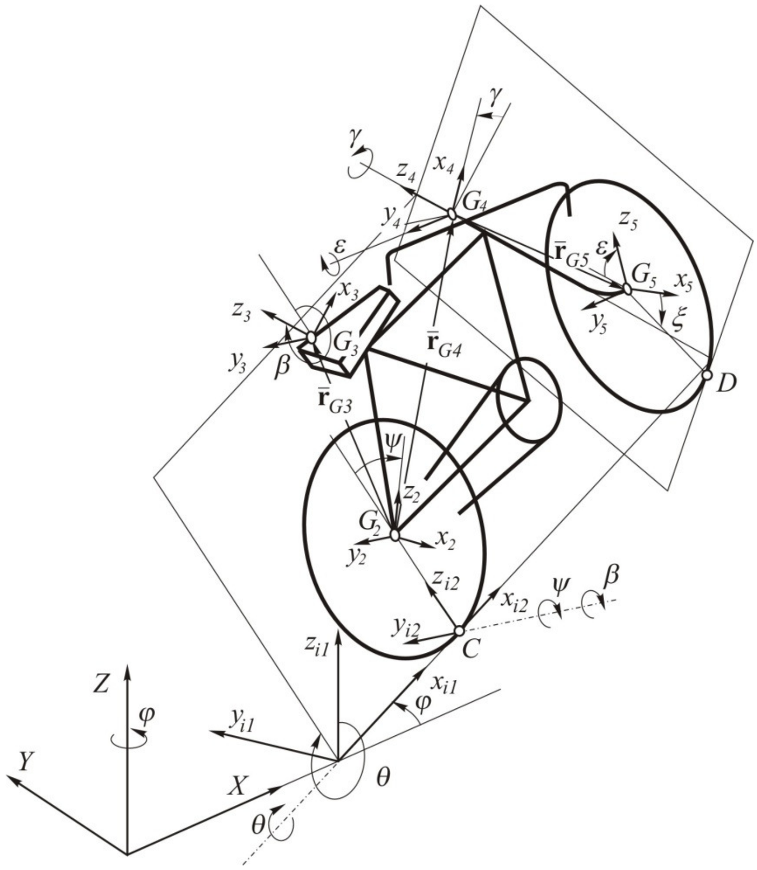

Figure 1. Bicycle in arbitrary position

The coordinates used to describe the position and orientation of the bicycle are:

1. Coordinates x_C and y_C of the real Wheel contact point *C* de in plane <*X Y*> of the global frame. In this work it is assumed that the bicycle advance in a flat plane with no inclination.
2. The heading angle (yaw) φ that form the axis x_{i1} of the first intermediate frame with axis *X* of the global frame.
3. The lean angle (roll) θ that forms axis z_{i2} of the second intermediate frame with axis z_{i1} of the first intermediate frame. Angles φ and θ determine the orientation of the plane that contains the frame of the bicycle.
4. The rolling angle (pitch) ψ that forms axis z_2 of the rear wheel with axis z_{i2} of the second intermediate frame.
5. Angle β that forms axis z_3 with axis z_{i2} of the second intermediate frame.
6. The steering angle γ that forms axis x_4 with axis x_3.
7. The rolling angle (pitch) ε that forms axis z_5 with axis z_4.

Finally, the coordinate set is grouped in the following vector:

$$\mathbf{q} = \begin{bmatrix} x_C & y_C & \varphi & \theta & \psi & \beta & \gamma & \varepsilon \end{bmatrix}^T \tag{1}$$

2.2 Velocities and angular velocities

In this section it is shown that the absolute velocity of the centre of gravity and the angular velocities of the bodies of the bicycle can be written in terms of the coordinate vector **q** and its time derivative $\dot{\mathbf{q}}$ as follows:

$$\mathbf{v}_{G_i} = \mathbf{H}^i(\mathbf{q})\,\dot{\mathbf{q}}, \quad \boldsymbol{\omega}^i = \mathbf{G}^i(\mathbf{q})\,\dot{\mathbf{q}} \tag{2}$$

where matrices $\mathbf{H}^i$ and $\mathbf{G}^i$ are functions of the position of the system. In what follows the calculation of these matrices for the rear wheel (body 2) is given as an example.

The position of the centre of gravity of the rear wheel is given by:

$$\mathbf{r}_{G_2} = \mathbf{r}_C + \mathbf{A}^{i2}\overline{\mathbf{r}}^{i2}_{G_2} = \begin{bmatrix} x_c \\ y_c \\ 0 \end{bmatrix} + \begin{bmatrix} \mathrm{sen}\varphi\mathrm{sen}\theta \\ -\cos\varphi\mathrm{sen}\theta \\ \cos\theta \end{bmatrix} R \tag{3}$$

and the velocity of the centre of gravity of the rear wheel is obtained as:

$$\mathbf{v}_{G_2} = \begin{bmatrix} \dot{x}_c \\ \dot{y}_c \\ 0 \end{bmatrix} + \begin{bmatrix} \cos\varphi\mathrm{sen}\theta\dot{\varphi} + \mathrm{sen}\varphi\cos\theta\dot{\theta} \\ \mathrm{sen}\varphi\mathrm{sen}\theta\dot{\varphi} - \cos\varphi\cos\theta\dot{\theta} \\ -\mathrm{sen}\theta\dot{\theta} \end{bmatrix} R = \mathbf{H}^2\dot{\mathbf{q}} \tag{3}$$

where matrix $\mathbf{H}^2$ is given by:

$$\mathbf{H}^2 = \begin{bmatrix} 1 & 0 & R\cos\varphi\mathrm{sen}\theta & R\mathrm{sen}\varphi\cos\theta & 0 & 0 & 0 & 0 \\ 0 & 1 & R\mathrm{sen}\varphi\mathrm{sen}\theta & -R\cos\varphi\cos\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -R\mathrm{sen}\theta & 0 & 0 & 0 & 0 \end{bmatrix} \tag{4}$$

For the symbolic calculation of matrices $\mathbf{H}^i$ it is interesting to note that this matrix is the jacobian of the position of the centre of gravity $\mathbf{r}_{Gi}$ with respect to vector **q**, this is $\mathbf{H}^i = \dfrac{\partial \mathbf{r}_{G_i}}{\partial \mathbf{q}}$.

The angular velocity of the rear wheel is obtained as:

$$\boldsymbol{\omega}^2 = \dot{\varphi}\mathbf{k} + \dot{\theta}\mathbf{i}^{i1} + \dot{\psi}\mathbf{j}^{i2} = \mathbf{G}^2\dot{\mathbf{q}} \tag{5}$$

where

$$\mathbf{G}^2(\mathbf{q}) = \begin{bmatrix} 0 & 0 & 0 & \cos\varphi & -\mathrm{sen}\varphi\cos\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{sen}\varphi & \cos\varphi\cos\theta & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \mathrm{sen}\theta & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & [\mathbf{A}^1]_3 & [\mathbf{A}^{i1}]_1 & [\mathbf{A}^{i2}]_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{6}$$

where $[\mathbf{A}^j]_i$ represents the i column of the orientation matrix $\mathbf{A}^j$. Accordingly, the angular velocity in the body frame can be obtained as follows:

$$\overline{\boldsymbol{\omega}}^2 = \dot{\varphi}\overline{\mathbf{k}} + \dot{\theta}\overline{\mathbf{i}}^{i1} + \dot{\psi}\overline{\mathbf{j}}^{i2} = \overline{\mathbf{G}}^2\dot{\mathbf{q}} \tag{7}$$

where matrix $\overline{\mathbf{G}}^2$, that is also a function of **q**, is given by:

$$\overline{\mathbf{G}}^2 = \begin{bmatrix} 0 & 0 & -\cos\theta \,\mathrm{sen}\,\psi & \cos\psi & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathrm{sen}\,\theta & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \cos\theta\cos\psi & \mathrm{sen}\,\psi & 0 & 0 & 0 & 0 \end{bmatrix} = \\ \begin{bmatrix} \mathbf{0} & \mathbf{0} & \left[\left(\mathbf{A}^2\right)^T\right]_3 & \left[\left(\mathbf{A}_\theta \mathbf{A}_\psi\right)^T\right]_1 & \left[\left(\mathbf{A}_\psi\right)^T\right]_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{8}$$

2.3 Constraints

The selected coordinates are not independent. They must fulfil a set of kinematic constraints. There are three types of constraints in this system:

1. *Contact constraints*. They guarantee that the front wheel has a contact point in the horizontal plane (due to the selected coordinates, contact constraints are not needed for the rear wheel). These contraints are *scleronomic*, because time does not appear explicitly, and *holonomic*, because generalized velocities do not appear.
2. *Rolling-without-sliding constraints*. These constraints guarantee that both wheels roll without sliding. These constrains are *non-holonomic* becacuse they are functions of **q** and $\dot{\mathbf{q}}$.
3. *Driving constraints*. This constraint guarantees the forward motion of the bicycle. It is assumed that the time derivative of ψ is constant, this is, coordinate ψ changes linearly with time. This is a *rehonomic* constraint because time appears explicitly.

The explicit form of the constraints is now given. Figure 2 shows two solids in (non-conformal) contact. This situation requires the vector of coordinates to fulfil two kind of constraints:

1. One point of solid *i*, the contact point *P*, is located in the same spatial position than other point of solid *j*.
2. The tangent plane to solid *i* at *P* has to be parallel to the tangent plane to solid *j* at *P*.

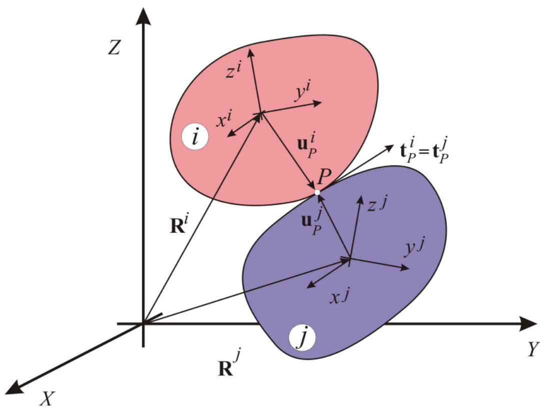

Figure 2. Solids in contact

The position of an arbitrary point of the front wheel is obtained in the body frame as:

$$\bar{\mathbf{r}}_P^5 = \begin{bmatrix} R\cos\xi & 0 & -R\mathrm{sen}\,\xi \end{bmatrix}^T, \quad \xi \in \begin{bmatrix} 0 & 2\pi \end{bmatrix} \tag{9}$$

where ξ is a geometric angular parameter. The tangent vector to the wheel at *P* in the body frame is obtained as:

$$\bar{\mathbf{t}}_P^5 = \frac{\partial \bar{\mathbf{r}}_P^5}{\partial \xi} = [-R\text{sen}\xi \quad 0 \quad R\cos\xi]^T, \quad \xi \in [0 \quad 2\pi] \tag{10}$$

The position and tangent vectors in the global frame are given by:

$$\mathbf{r}_P(\xi) = \mathbf{r}_{G_5} + \mathbf{A}^5 \bar{\mathbf{r}}_P^5, \quad \mathbf{t}_P(\xi) = \mathbf{A}^5 \bar{\mathbf{t}}_P^5, \quad \xi \in [0 \quad 2\pi] \tag{11}$$

In this particular case, in which the body in contact can be described by a line instead of a curve, the contact constrains are given by:

$$\left.\begin{matrix} [\mathbf{r}_D(\mathbf{q},\xi_D)]_Z = 0 \\ [\mathbf{t}_D(\mathbf{q},\xi_D)]_Z = 0 \end{matrix}\right\} \Rightarrow \mathbf{C}^{con}(\mathbf{q}) = \mathbf{0} \tag{12}$$

where D is the contact point at the front Wheel and ξ_D is the angular parameter associated with it. The two equations given in (12) are written in terms of vector $\mathbf{q}$ and the geometric parameter ξ_D that cannot be eliminated. That is why the contact constraints reduces one degree of freedom of the system (two equations an one new coordinate, 2 - 1 = 1). Therefore the new *non-generalized* coordinate ξ (for simplicity subscript D is eliminated in what follows) is introduced. The new vector of coordinates $\mathbf{p}$ is given by:

$$\mathbf{p} = [\mathbf{q}^T \quad \xi]^T = [x_C \quad y_C \quad \varphi \quad \theta \quad \psi \quad \beta \quad \gamma \quad \varepsilon \quad \xi]^T \tag{13}$$

Rolling-without-sliding constraints guarantee that the velocity of contact points of the wheels is zero. In the case of the rear wheel this velocity is given by:

$$\mathbf{v}_C = \mathbf{v}_{G_2} + \boldsymbol{\omega}^2 \wedge \mathbf{r}_{G_2 C} \tag{14}$$

where $\mathbf{r}_{G_2 C} = \mathbf{A}^{i2}[0 \quad 0 \quad -R]^T$. Equation (14) can be transformed to:

$$\mathbf{v}_C = \begin{bmatrix} \dot{x}_c - R\cos\varphi\dot{\psi} \\ \dot{y}_c - R\text{sen}\varphi\dot{\psi} \\ 0 \end{bmatrix}, \tag{15}$$

That can be reduced to:

$$\left.\begin{matrix} \dot{x}_c - R\cos\varphi\dot{\psi} = 0 \\ \dot{y}_c - R\text{sen}\varphi\dot{\psi} = 0 \end{matrix}\right\} \Rightarrow \mathbf{C}^{rod,2}(\mathbf{p},\dot{\mathbf{p}}) = \mathbf{0} \tag{16}$$

which can be written in the following matrix form:

$$\mathbf{B}^2(\mathbf{p})\,\dot{\mathbf{p}} = \mathbf{0}$$
$$\mathbf{B}^2 = \begin{bmatrix} 1 & 0 & 0 & 0 & -R\cos\varphi & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -R\text{sen}\varphi & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{17}$$

where matrix $\mathbf{B}^2$ can be obtained as the jacobian $\mathbf{B}^2 = \frac{\partial \mathbf{C}^{rod,2}}{\partial \dot{\mathbf{p}}}$. Fort the front wheel the constraints take the form:

$$\left.\begin{matrix} [\mathbf{v}_D]_X = 0 \\ [\mathbf{v}_D]_Y = 0 \end{matrix}\right\} \Rightarrow \mathbf{C}^{rod,5}(\mathbf{p},\dot{\mathbf{p}}) = \mathbf{0} \tag{18}$$

These equations can also be written as:

$$
\begin{aligned}
&\mathbf{B}^5(\mathbf{p})\,\dot{\mathbf{p}} = \mathbf{0} \\
&\mathbf{B}^5 = \frac{\partial \mathbf{C}^{rod,5}}{\partial \dot{\mathbf{p}}}
\end{aligned}
\tag{19}
$$

Finally the rolling-without-sliding constraints are given by:

$$
\mathbf{C}^{rod}(\mathbf{p},\dot{\mathbf{p}}) = \begin{bmatrix} \mathbf{C}^{rod,2}(\mathbf{p},\dot{\mathbf{p}}) \\ \mathbf{C}^{rod,5}(\mathbf{p},\dot{\mathbf{p}}) \end{bmatrix} = \mathbf{0}, \quad \mathbf{B}(\mathbf{p}) = \begin{bmatrix} \mathbf{B}^2(\mathbf{p}) \\ \mathbf{B}^5(\mathbf{p}) \end{bmatrix} \tag{20}
$$

The forward motion of the bicycle is imposed with the following driving constraint:

$$
\psi - \frac{V}{R}t = 0 \Rightarrow \mathbf{C}^{mov}(\mathbf{q},t) = 0\,, \tag{21}
$$

where V and R are the forward velocity of the bicycle and radius of the rear wheel, respectively.

The total set of constraint can be given with the vector:

$$
\mathbf{C}(\mathbf{p},\dot{\mathbf{p}},t) = \begin{bmatrix} \mathbf{C}^{con}(\mathbf{p}) \\ \mathbf{C}^{rod}(\mathbf{p},\dot{\mathbf{p}}) \\ \mathbf{C}^{mov}(\mathbf{p},t) \end{bmatrix} = \mathbf{0}, \tag{22}
$$

The kinematic description of the bicycle requires $\boldsymbol{n} = \mathbf{9}$ coordintates (8 in vector $\mathbf{q}$ plus parameter ξ) subjected to $\boldsymbol{m} = \mathbf{7}$ constraints (2 contact constraints, 4 rolling-without-sliding constraints and 1 mobility constraint). Therefore, the bicycle has $\boldsymbol{n} - \boldsymbol{m} = \mathbf{2}$ degrees of freedom.

3 DYNAMICS

The equations of motion of the system are obtained using Lagrange equations of the 1st kind. These equations are given by:

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\mathbf{p}}}\right) - \frac{\partial T}{\partial \mathbf{p}} + \mathbf{D}^T \boldsymbol{\lambda} = \mathbf{Q}_{grav} + \mathbf{Q}_{ext} \\
&\mathbf{C}(\mathbf{p},\dot{\mathbf{p}},t) = \mathbf{0}
\end{aligned}
\tag{23}
$$

where T is the kinetic energy, $\boldsymbol{\lambda}$ is the vector of multipliers, $\mathbf{Q}_{grav}$ is the vector of generalized gravity forces and $\mathbf{Q}_{ext}$ is the vector of generalized external forces. Matrix $\mathbf{D}$ is given by:

$$
\mathbf{D} = \begin{bmatrix} \mathbf{C}^{con}_{\mathbf{p}} \\ \mathbf{B} \\ \mathbf{C}^{mov}_{\mathbf{p}} \end{bmatrix} \tag{24}
$$

where $\mathbf{C}^i_{\mathbf{p}} = \dfrac{\partial \mathbf{C}^i}{\partial \mathbf{p}}$, $i = con, mov$, (jacobian of the holonomic constrains) and $\mathbf{B}$ is given in Eq. (20). Equation 23 is a system of differential-algebraic equations (DAE). This is the kind of equations of motion that appear in multibody dynamics [4].

3.1 Kinetic Energy

The kinetic energy of the bicycle is given by:

$$T = \sum_{i=2}^{5} \frac{1}{2} \left[m^i \left(\mathbf{v}_{G_i} \right)^T \mathbf{v}_{G_i} + \left(\overline{\boldsymbol{\omega}}^i \right)^T \overline{\mathbf{I}}^i \overline{\boldsymbol{\omega}}^i \right] \tag{25}$$

where m^i and $\overline{\mathbf{I}}^i$ are the mass and the inertia tensor of solid *i*, respectively. Introducing Eq. (2) into (25) yields

$$T = \sum_{i=2}^{5} \frac{1}{2} \left[m^i \dot{\mathbf{p}}^T \mathbf{H}^{i^T} \mathbf{H}^i \dot{\mathbf{p}} + \dot{\mathbf{p}}^T \overline{\mathbf{G}}^{i^T} \overline{\mathbf{I}}^i \overline{\mathbf{G}}^i \dot{\mathbf{p}} \right] = \sum_{i=2}^{5} \frac{1}{2} \dot{\mathbf{p}}^T \left[m^i \mathbf{H}^{i^T} \mathbf{H}^i + \overline{\mathbf{G}}^{i^T} \overline{\mathbf{I}}^i \overline{\mathbf{G}}^i \right] \dot{\mathbf{p}} \tag{26}$$

The kinetic energy can be written in compact form as:

$$T = \frac{1}{2} \dot{\mathbf{p}}^T \mathbf{M} \dot{\mathbf{p}}\,, \tag{27}$$

where the velocity-dependent mass matrix **M** is given by:

$$\mathbf{M}(\mathbf{p}) = \sum_{i=2}^{5} \left[m^i \mathbf{H}^{i^T} \mathbf{H}^i + \overline{\mathbf{G}}^{i^T} \overline{\mathbf{I}}^i \overline{\mathbf{G}}^i \right] \tag{28}$$

3.2 Inertia forces

The two first terms of the equations of motion (23) represent the inertia forces. They are given by:

$$\begin{gathered} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\mathbf{p}}} \right) = \mathbf{M}\ddot{\mathbf{p}} + \dot{\mathbf{M}}\dot{\mathbf{p}} \\ \frac{\partial T}{\partial \mathbf{p}} = \frac{1}{2} \frac{\partial [\mathbf{M}\dot{\mathbf{p}}]}{\partial \mathbf{p}} \dot{\mathbf{p}} \end{gathered} \tag{29}$$

These forces are the inertia forces that are proportional to the system accelerations ($\mathbf{M}\ddot{\mathbf{p}}$) plus the inertia forces that are quadratic with respect to the system velocities $\mathbf{Q}_v$ (centrifugal and Coriolis forces). The quadratic-velocity inertia terms are given by

$$\mathbf{Q}_v = -\dot{\mathbf{M}}\dot{\mathbf{p}} + \frac{\partial [\mathbf{M}\dot{\mathbf{p}}]}{\partial \mathbf{p}} \dot{\mathbf{p}} \tag{30}$$

3.3 Generalized gravity forces

In order to obtain the vecotr $\mathbf{Q}_{grav}$ in Eq. (23) the virtual power of the gravity forces is calculated:

$$\dot{W}_{grav} = \sum_{i=2}^{5} \mathbf{P}^{i^T} \mathbf{v}_{G_i}^* = \left(\sum_{i=2}^{5} \mathbf{P}^{i^T} \mathbf{H}^i \right) \dot{\mathbf{p}}^* \tag{31}$$

where $\mathbf{P}^i = \begin{bmatrix} 0 & 0 & -m_i g \end{bmatrix}^T$, is the weight of solid *i* and superscripts '*' means virtual magnitude. Vector $\mathbf{Q}_{grav}$ yields:

$$\mathbf{Q}_{grav} = \left(\sum_{i=2}^{5} \mathbf{H}^{i^T} \mathbf{P}^i \right) \tag{32}$$

The equations of motion finally take the form:

$$\begin{gathered} \mathbf{M}(\mathbf{p})\ddot{\mathbf{p}} + \mathbf{D}^T \boldsymbol{\lambda} = \mathbf{Q}_v(\mathbf{p}, \dot{\mathbf{p}}) + \mathbf{Q}_{grav}(\mathbf{p}) + \mathbf{Q}_{ext} \\ \mathbf{C}(\mathbf{p}, \dot{\mathbf{p}}, t) = \mathbf{0} \end{gathered} \tag{33}$$

4 EQUATIONS IN TERMS OF INDEPENDENT COORDINATES (DAE TO ODE)

The *generalized coordinate partitioning* method is now used to transform equations (33) to a ODE system written in terms of independent coordinates. The time derivative of the holonomic constraints together with the non-holonomic constraints represent a linear system of equations with respect to the system coordinates, as follows:

$$\left.\begin{aligned}\mathbf{C}_{\mathbf{p}}^{con}\dot{\mathbf{p}} &= \mathbf{0}\\ \mathbf{B}\dot{\mathbf{p}} &= \mathbf{0}\\ \mathbf{C}_{\mathbf{p}}^{mov}\dot{\mathbf{p}} &= V/R\end{aligned}\right\} \Rightarrow \begin{bmatrix}\mathbf{C}_{\mathbf{p}}^{con}\\ \mathbf{B}\\ \mathbf{C}_{\mathbf{p}}^{mov}\end{bmatrix}\dot{\mathbf{p}} = \begin{bmatrix}\mathbf{0}\\ \mathbf{0}\\ V/R\end{bmatrix} \Rightarrow \mathbf{D}\dot{\mathbf{p}} = -\mathbf{C}_t \tag{34}$$

where matrix $\mathbf{D}$ was defined in Eq. (24) and $\mathbf{C}_t$ is the partial derivative of the constraint equations with respect to time. Vector $\mathbf{p}$ is divided into independent coordinates $\mathbf{p}_i$, that contains the coordinates θ and γ, and the vector of dependent coordinates $\mathbf{p}_d$, that contains all other coordinates in the system. Equations (34) can be written in the form:

$$\begin{bmatrix}\mathbf{D}_i & \mathbf{D}_d\end{bmatrix}\begin{bmatrix}\dot{\mathbf{p}}_i\\ \dot{\mathbf{p}}_d\end{bmatrix} = -\mathbf{C}_t \tag{35}$$

Dependent velocities can be obtained as a function of the dependent velocities as follows:

$$\mathbf{D}_i\dot{\mathbf{p}}_i + \mathbf{D}_d\dot{\mathbf{p}}_d = -\mathbf{C}_t \;\Rightarrow\; \dot{\mathbf{p}}_d = \mathbf{D}_d^{-1}\left(-\mathbf{D}_i\dot{\mathbf{p}}_i - \mathbf{C}_t\right) \tag{36}$$

Therefore the system velocities can be obtained from the independent velocities as follows:

$$,\dot{\mathbf{p}} = \begin{bmatrix}\dot{\mathbf{p}}_i\\ \dot{\mathbf{p}}_d\end{bmatrix} = \begin{bmatrix}\mathbf{I}\\ -\mathbf{D}_d^{-1}\mathbf{D}_i\end{bmatrix}\dot{\mathbf{p}}_i + \begin{bmatrix}\mathbf{0}\\ -\mathbf{D}_d^{-1}\mathbf{C}_t\end{bmatrix} = \mathbf{E}\dot{\mathbf{p}}_i + \mathbf{F}, \tag{37}$$

where matrix $\mathbf{E}$ and vector $\mathbf{F}$ depend on the system coordinates and time. The system accelerations are obtained from the time derivative of Eq. (37):

$$\ddot{\mathbf{p}} = \mathbf{E}\ddot{\mathbf{p}}_i + \dot{\mathbf{E}}\dot{\mathbf{p}}_i + \dot{\mathbf{F}} = \mathbf{E}\ddot{\mathbf{p}}_i + \mathbf{J} \tag{38}$$

where $\mathbf{J} = \dot{\mathbf{E}}\dot{\mathbf{p}}_i + \dot{\mathbf{F}}$. Substituting (38) in the equations of motion and pre-multiplying by $\mathbf{E}^T$ yields:

$$\mathbf{E}^T\mathbf{M}\left(\mathbf{E}\ddot{\mathbf{p}}_i + \mathbf{J}\right) + \mathbf{E}^T\mathbf{D}^T\boldsymbol{\lambda} = \mathbf{E}^T\left(\mathbf{Q}_v + \mathbf{Q}_{grav} + \mathbf{Q}_{ext}\right) \tag{39}$$

It can be easily shown that the product $\mathbf{E}^T\mathbf{D}^T$ is zero. This property makes the Lagrange multipliers vector $\boldsymbol{\lambda}$ (reaction forces) to disappear from the new equations of motion. The new equations of motion yield:

$$\mathbf{E}^T\mathbf{M}\mathbf{E}\ddot{\mathbf{p}}_i = \mathbf{E}^T\left(\mathbf{Q}_v + \mathbf{Q}_{grav} + \mathbf{Q}_{ext} - \mathbf{M}\mathbf{J}\right) \tag{40}$$

Calling $\mathbf{M}_i = \mathbf{E}^T\mathbf{M}\mathbf{E}$ to the mass matrix in terms of the independent coordinates and $\mathbf{Q}_i = \mathbf{E}^T\left(\mathbf{Q}_v + \mathbf{Q}_{grav} + \mathbf{Q}_{ext} - \mathbf{M}\mathbf{J}\right)$ to the vector of applied forces yields:

$$\mathbf{M}_i\ddot{\mathbf{p}}_i = \mathbf{Q}_i \tag{41}$$

Equation (41) is a minimal set of ordinary differential equations. The system (33) is a set of $n + m = 9 + 7 = 16$ equations while the system (41) is a set of $n - m = 2$ equations of a simpler nature (ODE instead of DAE). However, it is important to notice that matrix $\mathbf{M}_i$ depends on the complete vector $\mathbf{p}$ and vector $\mathbf{Q}_i$ is a function of $\mathbf{p}$ and $\dot{\mathbf{p}}$. Therefore, the numerical solution of

(41) requires the solution of the non-linear constraint equations and their derivatives each time step. This step is not needed when solving the DAE system (33).

5 LINEARIZATION AND EIGENVALUE ANALYSIS AND STABILITY

Equations (41) can be linearized in the vicinity of a steady motion of the bicycle. In the upright position the generalized independent coordinates take the value $\theta = 0$ and $\gamma = 0$. Due to the forward velocity constraint V the system generalized velocities take the value $\dot{\psi} = \dot{\varepsilon} = V / R,\ \dot{\xi} = -V / R$. The values of coordinates x_c, y_c, φ, ψ and ε is irrelevant since they do not appear in the equations of motions. These coordinates are considered as *ignorable*. For linearization they can be considered as null. The upright steady motion of the bicycle is characterized by the following set of coordinates, velocities and accelerations:

$$
\begin{aligned}
\mathbf{p}_{std} &= \begin{bmatrix} x_C & y_C & \varphi & \theta & \psi & \beta & \gamma & \varepsilon & \xi \end{bmatrix}^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \beta_{std} & 0 & 0 & \xi_{std} \end{bmatrix}^T, \\
\dot{\mathbf{p}}_{std} &= \begin{bmatrix} \dot{x}_C & \dot{y}_C & \dot{\varphi} & \dot{\theta} & \dot{\psi} & \dot{\beta} & \dot{\gamma} & \dot{\varepsilon} & \dot{\xi} \end{bmatrix}^T = \begin{bmatrix} V & 0 & 0 & 0 & V/R & 0 & 0 & V/R & -V/R \end{bmatrix}^T, \\
\ddot{\mathbf{p}}_{std} &= \begin{bmatrix} \ddot{x}_C & \ddot{y}_C & \ddot{\varphi} & \ddot{\theta} & \ddot{\psi} & \ddot{\beta} & \ddot{\gamma} & \ddot{\varepsilon} & \ddot{\xi} \end{bmatrix}^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T
\end{aligned}
\tag{42}
$$

where β_{std} and ξ_{std} are obtained when solving the contact constraints. The linearized equations of motion are given by:

$$
\overline{\mathbf{M}}\ddot{\mathbf{p}}_i + \overline{\mathbf{C}}\dot{\mathbf{p}}_i + \overline{\mathbf{K}}\mathbf{p}_i = \mathbf{0}, \tag{43}
$$

where the mass matrix $\overline{\mathbf{M}}$, gyroscopic matrix $\overline{\mathbf{C}}$ and stiffness matrix $\overline{\mathbf{K}}$ are constants and given by:

$$
\overline{\mathbf{M}} = \mathbf{M}_i(\mathbf{p} = \mathbf{p}_{std}),\ \overline{\mathbf{C}} = \frac{\partial}{\partial \dot{\mathbf{p}}_i}(-\mathbf{Q}_i)(\mathbf{p} = \mathbf{p}_{std}, \dot{\mathbf{p}} = \dot{\mathbf{p}}_{std}),\ \overline{\mathbf{K}} = \frac{\partial}{\partial \mathbf{p}_i}(-\mathbf{Q}_i)(\mathbf{p} = \mathbf{p}_{std}, \dot{\mathbf{p}} = \dot{\mathbf{p}}_{std}) \tag{44}
$$

Once the equations (43) are obtained, the eigenvalue analysis is performed using the usual method for linear discrete system with non-proportional damping [Meirovitch] (complex modes). Equations (43) can be written in first order form, as follows:

$$
\begin{bmatrix} \dot{\mathbf{p}}_i \\ \ddot{\mathbf{p}}_i \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\overline{\mathbf{M}}^{-1}\overline{\mathbf{K}} & -\overline{\mathbf{M}}^{-1}\overline{\mathbf{C}} \end{bmatrix} \begin{bmatrix} \mathbf{p}_i \\ \dot{\mathbf{p}}_i \end{bmatrix} \Rightarrow \dot{\mathbf{y}} = \mathbf{A}\mathbf{y}, \qquad \mathbf{y} = \begin{bmatrix} \mathbf{p}_i \\ \dot{\mathbf{p}}_i \end{bmatrix} \tag{45}
$$

The eigenvalues and eigenvectors of matrix $\mathbf{A}$ characterize the bicycle dynamics for small motions around the steady motion considered. The eigenvalue analysis is performed as follows:

$$
\begin{aligned}
&\left|\mathbf{A} - \lambda\mathbf{I}\right| = 0 \ \Rightarrow\ \lambda_i,\ i = 1, 2, 3 \text{ y } 4 \\
&[\mathbf{A} - \lambda_i\mathbf{I}]\mathbf{\Phi}_i = \mathbf{0} \ \Rightarrow\ \mathbf{\Phi}_i
\end{aligned}
, \tag{46}
$$

where λ_i (frequencies and damping) are the eigenvalues and $\mathbf{\Phi}_i$ the eigenvectors (modes of vibration) of the bicycle.

6 SOLUTION PROCEDURE

The dynamics analysis presented in this paper has been implemented in Matlab using a combination of symbolic (up to Eq. (33)) and numerical computations. The transformation of the DAE system to ODE cannot be performed symbolically. The system parameters are those given in the benchmark presented in [5].

Figure 3 shows the four eigenvalues of the bicycle as a function of the forward velocity for a range between 0 and 10 m/s. For V between 0 and 0.6 m/s all eigenvalues are real, two positive and two negative valued. The bicycle is unstable. For V between 0.6 and 4.2 m/s two eigenvalues are negative and there are two complex conjugate eigenvalues with positive real part. The system is unstable but having an oscillatory behaviour. The dashed line shows the frequency of oscillation (imaginary part). For V between 4.2 and 6.8 m/s the real part of the complex conjugate eigenvalues is negative and the system becomes stable. Above 6.8 m/s there is a positive eigenvalue. Therefore the system turns unstable.

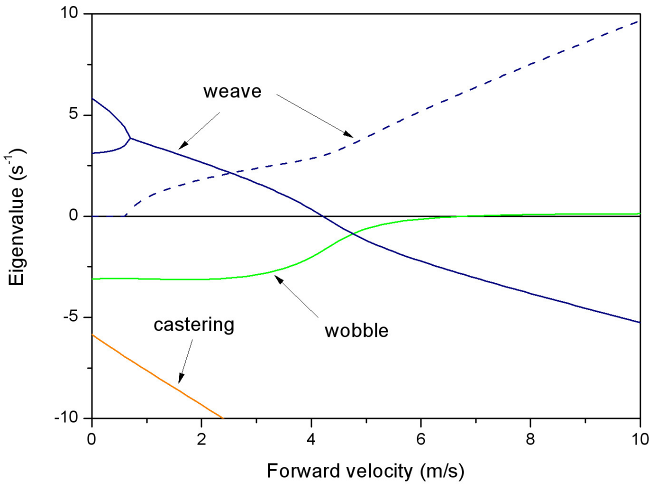

Figura 3. Bicycle eigenvalues as a function of the forward velocity

The bicycle is intrinsically stable only in the range between 4.2 and 6.8 m/s. Experience shows that the bicycle is unstable at low velocities. However, experience does not show instability at large velocities. The reason is that the real part of the eigenvalue associated with the unstable mode is very small. This means that the dynamics is very slow and the rider has plenty of time to stabilize the system by applying the necessary forces. It is important to notice that in the presented analysis the effect (control) of the rider has not been considered.

7 REAL-TIME INTERACTIVE SIMULATOR

The equations of motion of the bicycle are written in terms of generalized force vectors whose analytic expressions are known thanks to the symbolic computations. This property makes the numerical integration of the equations very efficient. The equations can be integrated into a real-time simulator that includes the signal from peripherals to virtually ride the bicycle. This section shows the implementation of the simulator and different alternatives for the bicycle control.

In the simulator the driving constraint $\mathbf{C}^{mov}(\mathbf{q},t)=0$ has been removed in order to accelerate and brake the bicycle. Therefore, the new model has $n - m = 9 - 6 = 3$ degrees of freedom.

7.1 Numerical integration

The numerical integration of the equations of motion follows the method described in [6]. The equations of motion of the bicycle are transformed into first order differential equations of the form:

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, t), \tag{47}$$

To this end, the coordinates vector is first divided into three groups, as follows:

$$\begin{gathered} \mathbf{p} = \begin{bmatrix} \mathbf{p}_{din}{}^T & \mathbf{p}_{kin}{}^T & \mathbf{p}_{dep}{}^T \end{bmatrix}^T \\ \mathbf{p}_{din} = \begin{bmatrix} \theta & \gamma & \psi \end{bmatrix}^T, \ \mathbf{p}_{kin} = \begin{bmatrix} x_c & y_c & \varphi & \varepsilon \end{bmatrix}^T \ \mathbf{p}_{dep} = \begin{bmatrix} \beta & \xi \end{bmatrix}^T \end{gathered} \tag{48}$$

and the vector of variables $\mathbf{y}$ is given by

$$\mathbf{y} = \begin{bmatrix} \mathbf{p}_{din} & \mathbf{p}_{kin} & \dot{\mathbf{p}}_{din} \end{bmatrix}^T, \tag{49}$$

The calculation of $\dot{\mathbf{y}}$ as a function of $\mathbf{y}$ and time t (implementation of function $\mathbf{f}$ given in Eq. 47) requires the following steps:

1. Calculation of vector $\mathbf{p}_{dep}$ by solving the two nonlinear contact constraint equations $\mathbf{C}^{con}(\mathbf{p}) = 0$.
2. Calculation of the system 6 velocities included in $\dot{\mathbf{p}}_{dep}$ and $\dot{\mathbf{p}}_{kin}$ by solving the linear algebraic equations $\mathbf{C}^{con}_{\mathbf{p}}\dot{\mathbf{p}} = \mathbf{0}$ (time derivative of the contact constraints) together with $\mathbf{B}\dot{\mathbf{p}} = \mathbf{0}$ (non-holonomic constraints).
3. Calculation of the terms of the DAE system (33) using symbolic expressions.
4. Transformation of the DAE system (33) to the ODE system (41) as described in Section 4. Notice that $\mathbf{p}_{din}$ takes de roll of the vector of independent coordinates $\mathbf{p}_i$ in Eq. (41) but including the angle ψ. $\dot{\mathbf{y}} = \begin{bmatrix} \dot{\mathbf{p}}_{din} & \dot{\mathbf{p}}_{kin} & \ddot{\mathbf{p}}_{din} \end{bmatrix}^T$
5. Calculation of the system accelerations $\ddot{\mathbf{p}}_{din}$ by solving Eq. 41. Vector of derivatives $\dot{\mathbf{y}} = \begin{bmatrix} \dot{\mathbf{p}}_{din} & \dot{\mathbf{p}}_{kin} & \ddot{\mathbf{p}}_{din} \end{bmatrix}^T$ is now fully known.

7.2 Control Inputs

The real-time simulator includes a number of input signals to ride the bicycle. The equations of motion are transformed to

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, \mathbf{u}, t), \tag{50}$$

where $\mathbf{u}$ is the vector of input signals. The bicycle control requires one input signal for the driving torque M_ψ (motor or braking torque) one signal for steering. The steering signal can be solved in two different ways:

1. Using the coordinate γ (angle of the steering bar) as input signal. In this case angle γ is treated as a guided coordinate and the system looses one degree of freedom.

2. Using a torque M_γ applied to the steering bar.

The two steering methods present problems in practise. Guiding angle γ directly does not work. The bicycle shows completely unpredictable motion. One physical reason and one mathematical reason can explain this fact. Physically, it is well known that in order to ride the bicycle one must not hold the steering bar too tightly. The handle bar must be free to adopt the angular position required by the system dynamics. Guiding the angle γ goes against this rule. The mathematical reason is that the system is unable to fulfil the rolling-without-sliding constraint at the front wheel at the time that it shows a *natural* motion.

Using a torque M_γ applied to the steering bar does not produce so bad results. When this method is implemented the simulator works and the bicycle shows what seems a natural behaviour. However the steering is unnatural. Assume that the peripheral used to control the bicycle looks like a steering bar and the input signal is proportional to the angle rotated by the steering bar. This is unnatural because the steering simply does not work like this. One can negotiate a curve adopting a non-zero angle γ while applying a zero torque M_γ. Actually this situation occurs in the hands-free steady curving of the bicycle. The use of torque M_γ as input signal shows that the implemented equations of motion correctly represent the bicycle dynamics because of the *counter-steering phenomenon*. The simulator shows that when the bicycle is doing a straight trajectory a short application of a torque M_γ to the right produces a final turn of the bicycle to the left.

In order to implement a different method to steer the bicycle, the upper body of the rider has been introduced into the system as a new solid. The motion of this new solid is described by the lean angle of the upper body η. However, this coordinate does not introduce a new degree of freedom because it is treated as a guided coordinate. Therefore, a third method is analysed to steer the bicycle:

3. Using the coordinate η (lean angle of the upper body) as input signal.

The control of the bicycle with this method provides relatively satisfactory results. The bicycle behaves as being driven by a rider with no hands at the steering bar. One can control the bicycle trajectory if no tight curves have to be negotiated.

Acknowledgments

Authors deeply acknowledge the help of Dr. Arend Schwab and Jodi Kooijman. Their advice has been essential to complete this work. This work was supported by the Spain's Ministry of Education and Science under project reference TRA2007-66808. This support is gratefully acknowledged.

References

[1] J.L. Escalona, R. Chamorro, "Stability analysis of vehicles on circular motions using multibody dynamics", *Nonlinear Dynamics* **53** (2008), pp. 237-250.

[2] R.A. Wehage and E.J. Haug, "Generalized coordinate partitioning for dimension reduction in analysis of constrained dynamic systems", *J. Mech. Des.* **134** (1982), pp. 247–255.

[3] J.D.G. Kooijman, A.L. Schwab and J.P. Meijaard, "Experimental validation of a model of an uncontrolled bicycle", *Multibody System Dynamics* **19** (2008), p.p. 115-132.

[4] A. A. Shabana, *Dynamics of multibody systems*, 3rd edition, 2005 (Cambridge University Press, Cambridge).

[5] J.P. Meijaard, J.M. Papadopoulos, A. Ruina, and A.L. Schwab, "Linearized dynamics equations for the balance and steer of a bicycle: a benchmark and review", *Proceedings of the Royal Society A*, **463 (**2007**)**, pp. 1955-1982.

[6] A.L. Schwab and J.P. Meijaard, "Dynamics of flexible multibody systems with non-holonomic constraints: a finite element approach", *Multibody System Dynamics* **10** (2003), p.p. 107-123.