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Automatic Generation of Linearised Equations of Motion for Moving Vehicles

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Abstract

This paper demonstrates a method for automatic generation of the linearised equations of motion for mechanical systems; in particular, one that is well suited to vehicle stability analysis. Unlike conventional methods for generating linearised equations of motion in 'MCK' form, the proposed method allows for the analysis of systems with nonholonomic constraints, and allows linearisation around non-zero speeds. With this method, the algebraic constraint equations are eliminated after the linearisation and reduction to first order. The method has been successfully applied to an assortment of problems of varying complexity.

The linearised unconstrained equations of motion, combined with the linearised kinematic differential equations, are given in Equation (1).

$$\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix} \begin{Bmatrix} \dot{\boldsymbol{p}} \\ \dot{\boldsymbol{w}} \end{Bmatrix} + \begin{bmatrix} \mathbf{V} & -\mathbf{I} \\ \mathbf{K} & \mathbf{C} \end{bmatrix} \begin{Bmatrix} \boldsymbol{p} \\ \boldsymbol{w} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \boldsymbol{f}_c + \boldsymbol{f}_\mathrm{a} \end{Bmatrix} \tag{1}$$

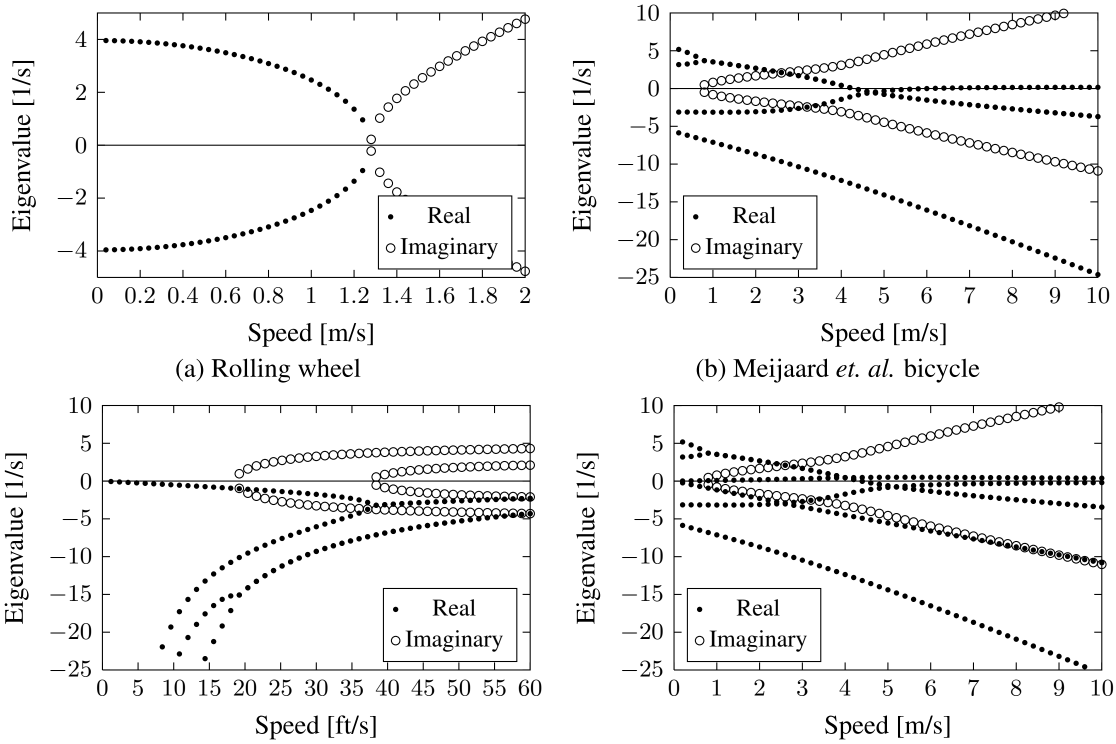
The $\mathbf{V}$ matrix results from the linearisation of the kinematic differential equations, and contains the skew symmetric matrix of the constant linear velocities of the bodies, arranged in the upper right 3x3 sub-matrix of the set of 6x6 matrices arranged along the diagonal. All other entries are zero. The $\mathbf{C}$ matrix contains the traditional damping matrix, plus terms due to the inertia forces, i.e., centripetal forces and gyroscopic moments. The stiffness matrix $\mathbf{K}$ is the sum of terms resulting from deflection of elastic elements and terms resulting from preloads in the system, i.e. the tangent stiffness matrix. The applied and constraint forces appear in the right hand side. The mass matrix $\mathbf{M}$ results from Newton's Laws, and is tri-diagonal as is typical.

The linearised constraint equations are written using a state vector combining both positions and velocities. The positions are expressed in a fixed global reference frame, where the velocities are given in a body fixed moving reference frame. When expressed in this form, identical coefficients describe the constraints as applied to global velocities and local accelerations. Because the positions and velocities are given as separate states, the holonomic constraint equations are applied twice; first to the positions, and again, in differentiated form, to the velocities. The nonholonomic constraints are applied only to the velocities. The combined constraint equations are given in Equation (2).

$$\begin{bmatrix} \mathbf{B}_\mathrm{h} & \mathbf{0} \\ -\mathbf{B}_\mathrm{h}\mathbf{V} & \mathbf{B}_\mathrm{h} \\ \mathbf{0} & \mathbf{B}_\mathrm{nh} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{p}} & \boldsymbol{p} \\ \dot{\boldsymbol{w}} & \boldsymbol{w} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{2}$$

The $\mathbf{B}_\mathrm{h}$ and $\mathbf{B}_\mathrm{nh}$ matrices represent the holonomic and nonholonomic constraint equations, respectively. An orthogonal complement matrix is used to eliminate the constraint equations and constraint forces, and similarly to define a new minimal system of coordinates. The method is explained in detail, and combined with a genetic search algorithm to find parameters that stabilise a narrow tilting vehicle in [1]. The method has been implemented in the MATLAB$^{\mathrm{TM}}$/Octave programming language, under the title 'EoM', and is freely available under the GPL licence on the author's website.

The results produced have been verified against a number of benchmark problems from the literature, such as the rolling wheel (r=0.5 m) illustrated in Greenwood[2], the Meijaard *et. al.* rigid-rider bicycle[3], and the Ellis truck and trailer[4], as shown in Figures 1a, 1b, and 1c, respectively. More recently, the method has been applied to a bicycle and trailer combination; results are shown in Figure 1d.



(a) Rolling wheel

(b) Meijaard *et. al.* bicycle

(c) Truck and trailer

(d) Bicycle with trailer

Figure 1: Eigenvalues vs. speed

The bicycle used with the trailer for the model was the previously mentioned benchmark, and the trailer was modelled as a rigid body, attached to the bike by spherical joint, rolling on two wheels, each identical to the rear wheel of the bicycle. The properties of trailer are given in Table 1, using the coordinate system from the benchmark.

Table 1: Trailer parameters

mass	15 [kg]	centre of mass	-0.75,0,-0.4 [m]	tow hitch	0,0,-0.3 [m]
I_{xx}, I_{yy}, I_{zz}	1,1,3 [kg·m^2]	left wheel	-0.9,-0.3,-0.3 [m]		
I_{xy}, I_{yz}, I_{zx}	0,0,0 [kg·m^2]	right wheel	-0.9,0.3,-0.3 [m]		

References

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